Dynamics and Real-Time Simulation (DARTS) Laboratory

Spatial Operator Algebra (SOA)

12. Flexible Body Dynamics

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https://dartslab.jpl.nasa.gov/

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SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

See https://dartslab.jpl.nasa.gov/References/index.php for publications and references on the SOA methodology.

SOA Generalization Track Topics

8. **Graph theory based structure** – BWA matrices, connection to multibody systems
9. **Multibody graph systems** – generalization to tree topology rigid body systems, SKO/SPO operators, gather/scatter algorithms
10. **Closed-chain dynamics (cut-joint)** – holonomic and non-holonomic constraints, cut-joint method, operational space inertias, projected dynamics
11. **Closed-chain dynamics (constraint embedding)** – multibody graph transformations, constraint embedding for graph transformation, minimal coordinate closed-chain dynamics
12. **Flexible body dynamics** – Extension to flexible bodies, modal representations, recursive flexible body dynamics

Recap

Previous Session Recap

- Developed notions of graph partitioning
- Applied these to partitioning SKO models
- Defined conditions for partitioning to preserve tree structure
- Developed notion of subgraph aggregation
- Derived SKO model for aggregated graph
- Built constraint embedding idea on notion of subgraph aggregation
- Developed SKO model for closed-loop systems using constraint embedding

Comments

- Observations on the SKO model for constraint embedding:
  - The SKO operator elements are not 6x6 (for aggregated bodies)
  - The elements are not square or invertible
  - The elements size can vary from row to row

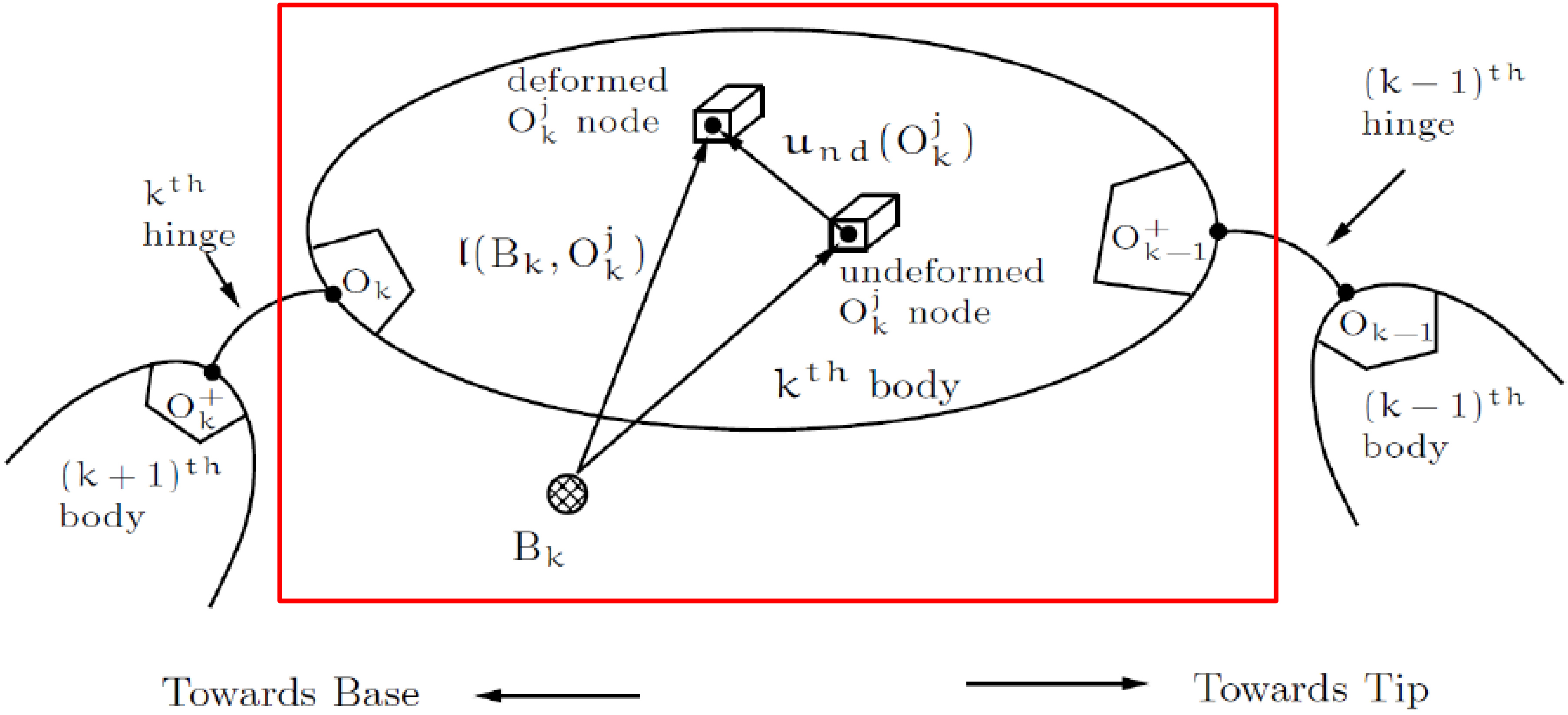
Single Flexible Body: Nodal Model

Flexible bodies

- So far we have focused on multibody systems with rigid bodies
- Rigidity is an idealization, and often bodies can have non-negligible deformation that needs to be included in the dynamics model
- We focus here on extending our development to lumped models for flexible bodies undergoing small deformation
- Our goal is to develop an SKO model for such flexible body systems
  - Once we have an SKO model, all the associated analysis and recursive algorithms will follow

A typical flexible body – nodal model

Think of a flexible body as a collection of rigid nodes (often point masses) connected by springs.



Use a floating frame of reference for the body

Nodal equations of motion (node frame)

SIngle node $\mathcal{O}_k^j$ properties

$\mathcal{O}_k^j$:   jth node on kth body

Each node can undergo translational and rotational deformations

$$\mathfrak{l}(k, \mathcal{O}_k^j) = \mathfrak{l}_0(k, \mathcal{O}_k^j) + \delta_\mathfrak{l}(\mathcal{O}_k^j) \in \mathcal{R}^3$$

Rotational deformation

$$\Delta_r(\mathcal{O}_k^j) \triangleq \begin{pmatrix} \delta_r(\mathcal{O}_k^j) & 0 \\ 0 & \delta_r(\mathcal{O}_k^j) \end{pmatrix} \in \mathcal{R}^{6\times 6}$$

Node spatial inertia

A node's spatial inertia (in local node frame)

$$M_{nd}(O_k^j) = \begin{pmatrix} \mathscr{J}(O_k^j) & m(O_k^j)\tilde{p}(O_k^j) \\ -m(O_k^j)\tilde{p}(O_k^j) & m(O_k^j)I_3 \end{pmatrix} \in \mathcal{R}^{6\times 6}$$

The node's spatial inertia (in body frame)

$$\underline{M}_{nd}(O_k^j) = \Delta_r(O_k^j) M_{nd}(O_k^j) \Delta_r^*(O_k^j) \in \mathcal{R}^{6\times 6} = \begin{pmatrix} \underline{\mathscr{J}}(O_k^j) & m(O_k^j)\tilde{\underline{p}}(O_k^j) \\ -m(O_k^j)\tilde{\underline{p}}(O_k^j) & m(O_k^j)I_3 \end{pmatrix}$$

Flexible body node velocity kinematics

body frame's inertial spatial velocity

$$\mathcal{V}(k) = \begin{bmatrix} \omega(k) \\ v(k) \end{bmatrix} \in \mathcal{R}^6.$$

rigid body transformation matrix for the node

$$\phi(k, O_k^j) \triangleq \begin{pmatrix} I & \tilde{l}(k, O_k^j) \\ 0 & I \end{pmatrix} \in \mathcal{R}^{6\times 6}$$

$$\mathcal{V}(O_k^j) = \phi^*(k, O_k^j)\mathcal{V}(k) + \delta_{nd}^{\mathcal{V}}(O_k^j) \quad \in \mathcal{R}^6$$

nodal inertial spatial velocity

$$\delta_{nd}^{\mathcal{V}}(O_k^j) = \begin{bmatrix} \delta_\omega(O_k^j) \\ \delta_v(O_k^j) \end{bmatrix} \in \mathcal{R}^6$$

nodal deformation spatial velocity

Individual node equations of motion

Standard rigid body equations of motion for a single node in the nodal frame

$$\alpha_{nd}(O_k^j) = \Delta_r(O_k^j)\,\frac{d}{dt}\left[\Delta_r^*(O_k^j)\,\mathcal{V}(O_k^j)\right]$$

nodal spatial accel

$$\mathfrak{f}_{nd}(O_k^j) = \underline{M}_{nd}(O_k^j)\alpha_{nd}(O_k^j) + \mathfrak{b}(O_k^j) + \mathfrak{f}_{nd}^{st}(O_k^j)$$

inter-node elastic deformation spatial force

$$\mathfrak{b}(O_k^j) = \overline{\mathcal{V}}(O_k^j)\underline{M}_{nd}(O_k^j)\mathcal{V}(O_k^j)$$

nodal gyroscopic spatial force

At this point the node equations of motion are in their own local (and different) frames

Nodal equations of motion (body frame)

Common body frame

- Instead of working with equations of motion with respect to individual node frames, want equations of motion wrt a common frame

- We will do so wrt the body floating frame

Nodal spatial acceleration expression

$$\alpha_{nd}(O^j_k) = \Delta_r(O^j_k)\,\frac{d}{dt}\left[\Delta^*_r(O^j_k)\,\mathcal{V}(O^j_k)\right]$$

body frame spatial accel

nodal deformation spatial accel

$$\alpha_{nd}(O^j_k) = \phi^*(k, O^j_k)\alpha(k) + \mathring{\delta}^{\mathcal{V}}_{nd}(O^j_k) + \mathfrak{a}_f(k, O^j_k)$$

nodal spatial accel in terms of body spatial accel

$$\mathfrak{a}_f(k, O^j_k) \triangleq \tilde{\mathcal{V}}(O^j_k)\delta^{\mathcal{V}}_{nd}(O^j_k) - \begin{bmatrix} 0 \\ \tilde{\delta}_\omega(O^j_k)\delta_v(O^j_k) \end{bmatrix} = \begin{bmatrix} \tilde{\omega}(k)\delta_\omega(O^j_k) \\ \tilde{\omega}(k)\delta_v(O^j_k) + \tilde{v}(O^j_k)\delta_\omega(O^j_k) \end{bmatrix}$$

Coriolis accel term

Nodal equations of motion using common body frame

Had
$$\mathfrak{f}_{nd}(\mathbb{O}_k^j) = \underline{\mathcal{M}}_{nd}(\mathbb{O}_k^j)\alpha_{nd}(\mathbb{O}_k^j) + \mathfrak{b}(\mathbb{O}_k^j) + \mathfrak{f}_{nd}^{st}(\mathbb{O}_k^j)$$

Use
$$\alpha_{nd}(\mathbb{O}_k^j) = \phi^*(k, \mathbb{O}_k^j)\alpha(k) + \mathring{\delta}_{nd}^{\mathcal{V}}(\mathbb{O}_k^j) + \mathfrak{a}_f(k, \mathbb{O}_k^j)$$

to get transformed equations of motion

$$\mathfrak{f}_{nd}(\mathbb{O}_k^j) \overset{B.1.14, B.1.9}{=} \underline{\mathcal{M}}_{nd}(\mathbb{O}_k^j)\left[\phi^*(k, \mathbb{O}_k^j)\alpha(k) + \mathring{\delta}_{nd}^{\mathcal{V}}(\mathbb{O}_k^j)\right] + \mathfrak{Q}(\mathbb{O}_k^j) + \mathfrak{f}_{nd}^{st}(\mathbb{O}_k^j)$$

$$\mathfrak{Q}(\mathbb{O}_k^j) \triangleq \underline{\mathcal{M}}_{nd}(\mathbb{O}_k^j)\,\mathfrak{a}_f(k, \mathbb{O}_k^j) + \mathfrak{b}(\mathbb{O}_k^j) = \begin{bmatrix} \tilde{\omega}(\mathbb{O}_k^j)\underline{\mathcal{J}}(\mathbb{O}_k^j)\omega(\mathbb{O}_k^j) + \underline{\mathcal{J}}(\mathbb{O}_k^j)\tilde{\omega}(k)\delta_\omega(\mathbb{O}_k^j) \\ +\mathfrak{m}(\mathbb{O}_k^j)\underline{\tilde{p}}(\mathbb{O}_k^j)\tilde{\omega}(k)\left(v(\mathbb{O}_k^j) + \delta_v(\mathbb{O}_k^j)\right) \\ \hline \mathfrak{m}(\mathbb{O}_k^j)\left[-\underline{\tilde{p}}(\mathbb{O}_k^j)\tilde{\omega}(k)\delta_\omega(\mathbb{O}_k^j) - \tilde{\omega}(\mathbb{O}_k^j)\underline{\tilde{p}}(\mathbb{O}_k^j)\omega(\mathbb{O}_k^j) \right. \\ \left. +\tilde{\omega}(k)\left(v(\mathbb{O}_k^j) + \delta_v(\mathbb{O}_k^j)\right)\right] \end{bmatrix}$$

velocity dependent spatial force

Modal representation at the node level

Modal representation

- A small deformation assumption, allows us to use linear expressions for rotational deformations
- The linearity allows us to use ‘modes’ as an alternative way to describe the deformation of the nodes on the body
- Modes are very useful since truncation can be used to develop reduced order models – especially for control system development

Modal expansion

Small rotational deformation linearity assumption

$$\delta_r(O_k^j) \triangleq \exp\left[\tilde{\delta}_q(O_k^j)\right] \approx I_3 + \tilde{\delta}_q(O_k^j) \in \mathcal{R}^{3\times 3}$$

small rotational deformation linear approximation

modal coordinates

$$u_{nd}(O_k^j) \overset{B.1.18}{=} \begin{bmatrix} \delta_q(O_k^j) \\ \delta_l(O_k^j) \end{bmatrix} = \sum_{r=1}^{n_{md}(k)} \Pi_r(O_k^j)\, \eta_r(k)$$

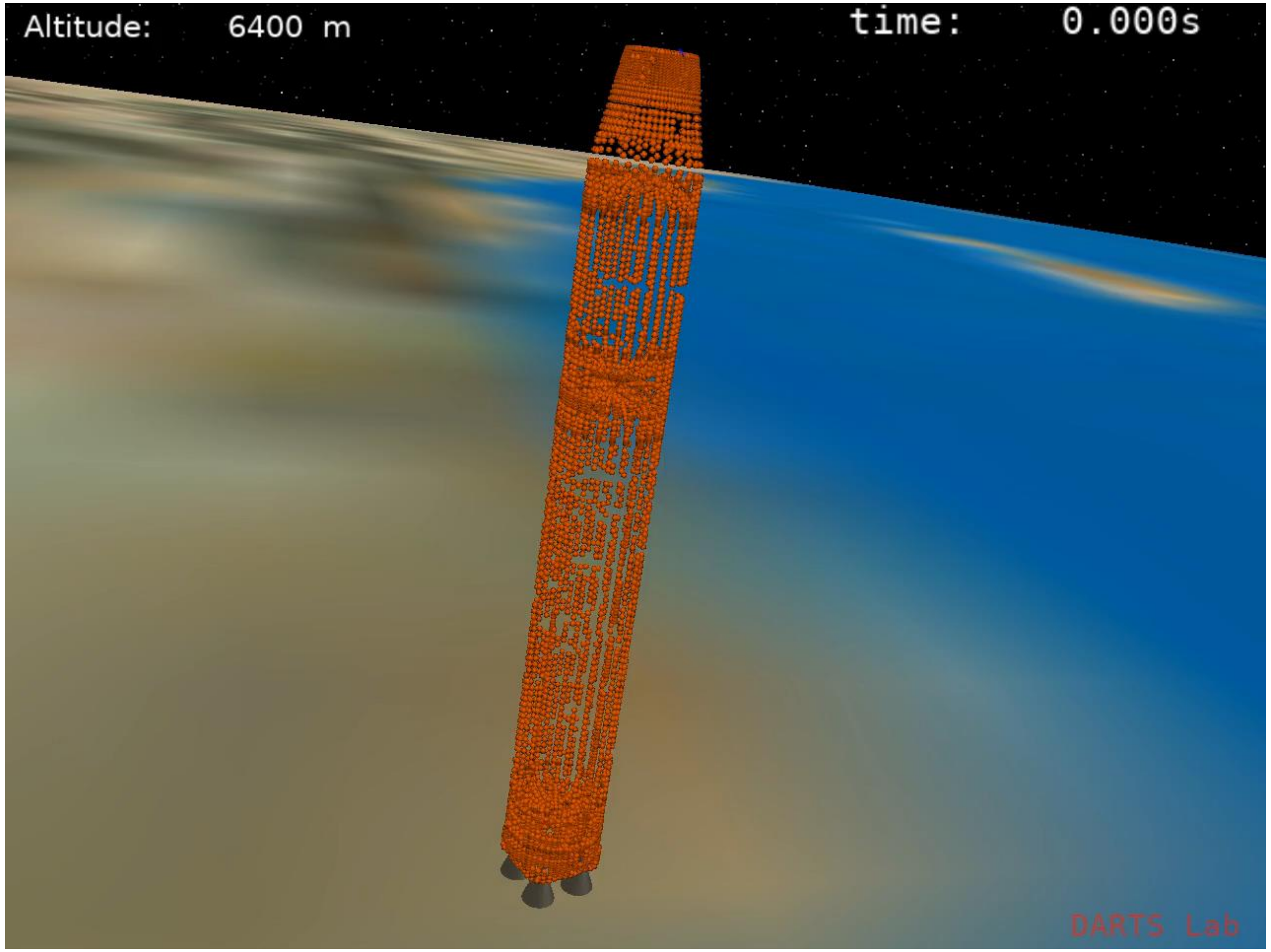
modal expansion of node deformation

$$\Pi_r(O_k^j) \triangleq \begin{pmatrix} \lambda_r^j(k) \\ \gamma_r^j(k) \end{pmatrix} \in \mathcal{R}^6$$

modal influence vector for the rth mode at the jth node

Bending mode example

Launch vehicle example



Matrix form reexpression

$$u_{nd}(O_k^j) = \Pi(O_k^j)\eta(k)$$

modal coordinates to jth node deformation mapping

modal influence matrix for the jth node

$$\Pi(O_k^j) \triangleq \left[\Pi_1(O_k^j), \cdots, \Pi_{n_{md}(k)}(O_k^j)\right]_{r=1}^{n_{md}(k)} \in \mathcal{R}^{6\times n_{md}(k)}$$

$$\delta^{\mathcal{V}}_{nd}(O_k^j) \overset{B.1.20,B.1.8}{=} \Pi(O_k^j)\dot{\eta}(k)$$

deformation velocity level mapping

Modal representation at the body level

Stacked vector across body nodes

$$\mathfrak{u}_{nd}(k) \triangleq \text{col}\left\{\mathfrak{u}_{nd}(O_k^j)\right\}_{j=1}^{n_{nd}(k)} \in \mathcal{R}^{6n_{nd}(k)}$$

$$\delta^{\mathcal{V}}_{nd}(k) \triangleq \text{col}\left\{\delta^{\mathcal{V}}_{nd}(O_k^j)\right\}_{j=1}^{n_{nd}(k)} \in \mathcal{R}^{6n_{nd}(k)}$$

$$\Pi(k) \triangleq \text{col}\left\{\Pi(O_k^j)\right\}_{j=1}^{n_{nd}(k)} \in \mathcal{R}^{6n_{nd}(k) \times n_{md}(k)}$$

$$\mathfrak{u}_{nd}(k) \overset{B.1.20}{=} \Pi(k)\eta(k)$$

$$\delta^{\mathcal{V}}_{nd}(k) = \Pi(k)\dot{\eta}(k)$$

stacked vector deformation expressions

All node velocities

$$\mathcal{V}(\mathbb{O}_k^j) = \phi^*(k, \mathbb{O}_k^j)\mathcal{V}(k) + \delta^{\mathcal{V}}_{nd}(\mathbb{O}_k^j)$$

$$\mathcal{V}_{nd}(k) \triangleq \text{col}\left\{\mathcal{V}\left(\mathbb{O}_k^j\right)\right\}_{j=1}^{n_{nd}(k)}$$

stacked vector of nodal spatial velocities

$$\mathcal{B}(k) \triangleq \left[\phi\left(k, \mathbb{O}_k^1\right), \phi\left(k, \mathbb{O}_k^2\right), \cdots, \phi\left(k, \mathbb{O}_k^{n_{nd}(k)}\right)\right] \in \mathcal{R}^{6\times 6n_{nd}(k)}$$

$$\mathcal{V}_{nd}(k) = \mathcal{B}^*(k)\mathcal{V}(k) + \delta^{\mathcal{V}}_{nd}(k)$$

stacked vector of nodal spatial velocities from body and deformation spatial velocities

Velocity for a flexible body

- For a rigid body, the spatial velocity serves as its body velocity
- For a flexible body, we augment it with the deformation velocity coordinates

$$\mathcal{V}_{fl}(k) \triangleq \begin{bmatrix} \dot{\eta}(k) \\ \mathcal{V}(k) \end{bmatrix} \in \mathcal{R}^{\check{N}(k)}$$

- Mapping from body velocity to nodal velocities

$$\mathcal{V}_{nd}(k) = Y(k)\mathcal{V}_{fl}(k) \in \mathcal{R}^{6n_{nd}(k)}$$

$$Y(k) \triangleq [\Pi(k), \ \mathcal{B}^*(k)] \in \mathcal{R}^{6n_{nd}(k)\times\check{N}(k)}$$

Body kinetic energy expression

$$\underline{M}_{nd}(k) \triangleq \text{diag}\left\{\underline{M}_{nd}(O_k^j)\right\} \in \mathcal{R}^{6n_{nd}(k)\times 6n_{nd}(k)}$$

$$M_{fl}(k) = Y^*(k)\underline{M}_{nd}(k)Y(k) \in \mathcal{R}^{\check{N}(k)\times\check{N}(k)}$$

$$\mathcal{V}_{nd}(k) = Y(k)\mathcal{V}_{fl}(k) \in \mathcal{R}^{6n_{nd}(k)}$$

$$\mathfrak{K}_e(k) = \frac{1}{2}\sum_{j=1}^{n_{nd}(k)} \mathcal{V}^*\left(\mathbb{O}_j^j\right) M_{nd}(\mathbb{O}_k^j)\mathcal{V}\left(\mathbb{O}_j^j\right) \overset{14.13,14.18}{=} \frac{1}{2}\mathcal{V}_{nd}^*(k)M_{nd}(k)\mathcal{V}_{nd}(k)$$

$$= \frac{1}{2}\mathcal{V}_{fl}^*(k)M_{fl}(k)\mathcal{V}_{fl}(k)$$

Flexible body equations of motion

Gathering together the individual node equations of motion, we have

$$\mathfrak{T}_{fl}(k) \triangleq Y^*(k)\,\mathrm{col}\left\{\mathfrak{f}_{nd}(O_k^j)\right\}_{j=1}^{n_{nd}(k)} \overset{B.1.34,B.1.15}{=} \mathcal{M}_{fl}(k)\alpha_{fl}(k) + \mathfrak{b}_{fl}(k) + Y^*\mathfrak{f}^{st}_{nd}(O_k^j)$$

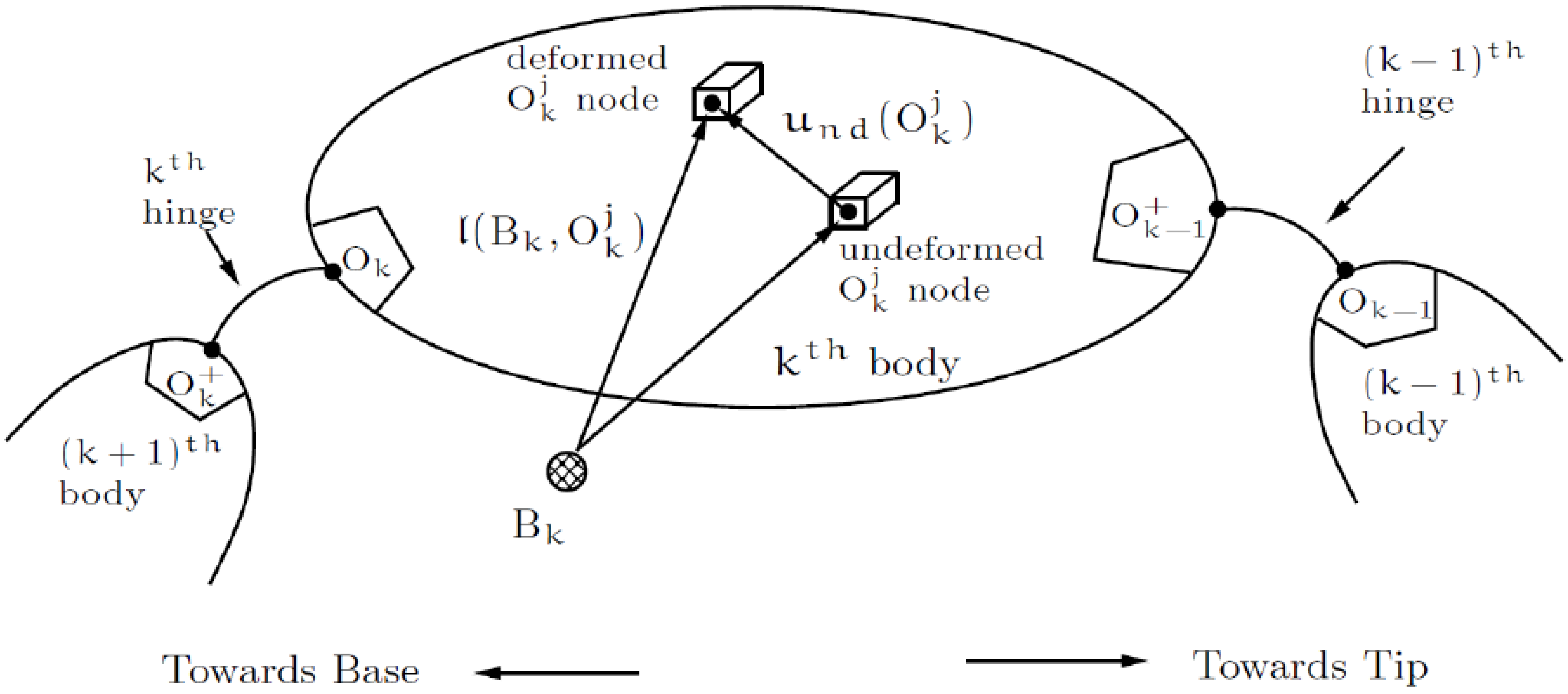
$$\mathfrak{b}_{fl}(k) \triangleq Y^* Q(O_k^j) \overset{B.1.16}{=} Y^*\left[\underline{M}_{nd}(O_k^j)\,\mathfrak{a}_f(k, O_k^j) + \mathfrak{b}(O_k^j)\right]$$

Modal integrals can be computed offline and used to simplify the computation of the terms in these equations of motion

Back to multibody system

Coupled flexible bodies

Large articulation, small deformation



Towards an SKO model

- Assume serial chain for notational simplicity
- A key step in the development of an SKO model is identifying its SKO operator
- The elements of the SKO operator are defined by the coefficient matrices involved in a recursive velocity relationship from parent to child body
- For rigid bodies this took the form

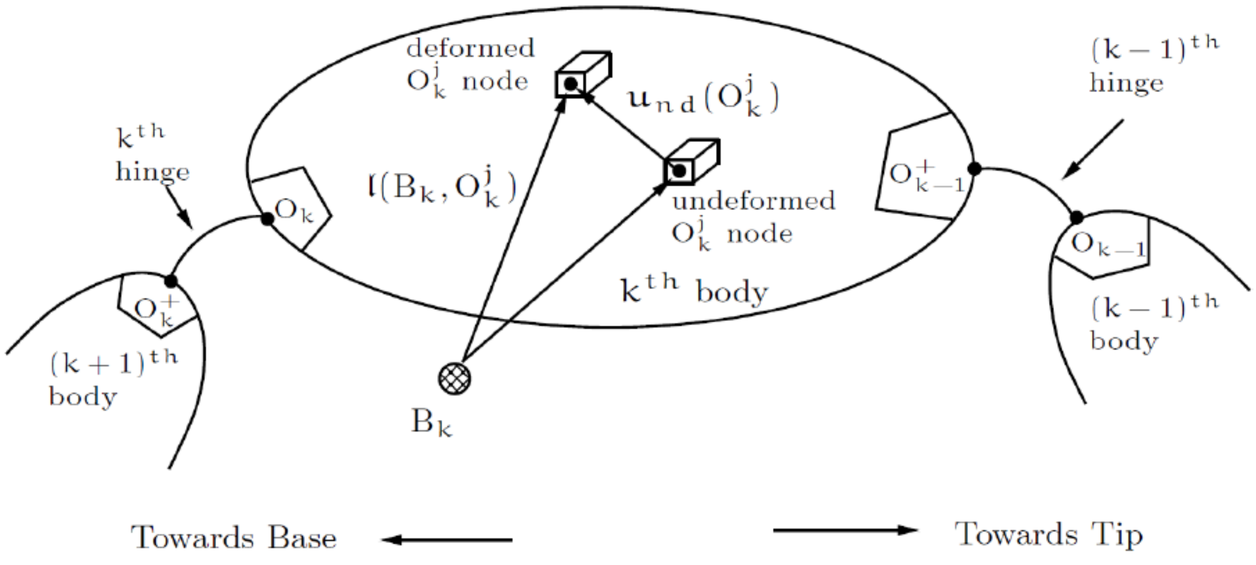
$$\mathcal{V}(k) = \mathbb{A}^*(\wp(k), k)\mathcal{V}(\wp(k)) + H^*(k)\dot{\theta}(k)$$

- Want to extend this to flexible bodies

Expression for body frame spatial velocity in terms of parent velocities

flexible child body velocity

$$\mathcal{V}_{fl}(k) \triangleq \begin{bmatrix} \dot{\eta}(k) \\ \mathcal{V}(k) \end{bmatrix}$$



parent body rigid body motion contribution

parent body deformation contribution

$$\mathcal{V}(k) = \phi^*(k+1,k)\mathcal{V}(k+1) + \phi^*(\mathbb{O}_k^+,k)\Pi(\mathbb{O}_{k+1}^+)\dot{\eta}(k+1) + \phi^*(\mathbb{O}_k,k)\left[H^*(k)\dot{\theta}(k) - \Pi(\mathbb{O}_k)\dot{\eta}(k)\right]$$

hinge articulation contribution

child body deformation contribution

Recursive velocity expression

$$\mathcal{V}(k) = \phi^*(k+1,k)\mathcal{V}(k+1) + \phi^*(\mathbb{O}_k^+,k)\Pi(\mathbb{O}_{k+1}^+)\dot{\eta}(k+1) + \phi^*(\mathbb{O}_k,k)\left[H^*(k)\dot{\theta}(k) - \Pi(\mathbb{O}_k)\dot{\eta}(k)\right]$$

$$\mathcal{V}_{fl}(k) \triangleq \begin{bmatrix} \dot{\eta}(k) \\ \mathcal{V}(k) \end{bmatrix}$$

$$\mathcal{V}_{fl}(k) \overset{14.39,14.36}{=} \Phi_{fl}^*(k+1,k)\mathcal{V}_{fl}(k+1) + H_{fl}^*(k)\dot{\vartheta}(k)$$

$$\vartheta(k) \triangleq \begin{bmatrix} \eta(k) \\ \theta(k) \end{bmatrix}$$

$$\Phi_{fl}(k+1,k) \triangleq \begin{pmatrix} \mathbf{0} & \Pi^*(\mathbb{O}_{k+1}^+)\phi(\mathbb{O}_k^+,k) \\ \mathbf{0} & \phi(k+1,k) \end{pmatrix}$$

generalized inter-body transformation matrix

$$H_{fl}(k) \triangleq \begin{pmatrix} \mathbf{I} & -\Pi_{\mathbb{B}}^*(\mathbb{O}_k) \\ \mathbf{0} & H_{\mathbb{B}}(k) \end{pmatrix}$$

generalized joint map matrix

$$H_{\mathbb{B}}(k) \triangleq H(k)\phi(\mathbb{O}_k,k) \in \mathcal{R}^{r_\nu(k)\times 6}$$

$$\Pi_{\mathbb{B}}(\mathbb{O}_k) \triangleq \phi^*(\mathbb{O}_k,k)\Pi(\mathbb{O}_k) \in \mathcal{R}^{6\times\overline{\mathcal{N}}(k)}$$

Comments

$$\mathcal{V}_{\mathfrak{fl}}(k) \overset{14.39,14.36}{=} \Phi^*_{\mathfrak{fl}}(k+1,k)\mathcal{V}_{\mathfrak{fl}}(k+1) + H^*_{\mathfrak{fl}}(k)\dot{\vartheta}(k)$$

- This is the recursive velocity expression we are looking for that helps us identify the elements of the SKO operator
- The elements row-size 6+the number of modes in the body
- Hence the row-size can vary from body to body

SKO/SPO operators for serial-chain of flexible bodies

$$\mathcal{E}_{\Phi_{fl}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Phi_{fl}(2,1) & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Phi_{fl}(3,2) & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \Phi_{fl}(n,n-1) & \mathbf{0} \end{pmatrix}$$

SKO operator

$$\Phi_{fl} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \Phi_{fl}(2,1) & \mathbf{I} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{fl}(n,1) & \Phi_{fl}(n,2) & \ldots & \mathbf{I} \end{pmatrix}$$

SPO operator

SKO model equations of motion

$$\mathcal{V}_{\mathfrak{fl}} = \Phi^*_{\mathfrak{fl}} H^*_{\mathfrak{fl}} \dot{\boldsymbol{\vartheta}}$$

$$\alpha_{\mathfrak{fl}} = \Phi^*_{\mathfrak{fl}} (H^*_{\mathfrak{fl}} \ddot{\boldsymbol{\vartheta}} + \mathfrak{a}_{\mathfrak{fl}})$$

$$\mathfrak{f}_{\mathfrak{fl}} = \Phi_{\mathfrak{fl}} (M_{\mathfrak{fl}} \alpha_{\mathfrak{fl}} + \mathfrak{b}_{\mathfrak{fl}} + \mathfrak{K}\vartheta)$$

$$\mathcal{T}_{\mathfrak{fl}} = H_{\mathfrak{fl}} \mathfrak{f}_{\mathfrak{fl}} = \mathcal{M}_{\mathfrak{fl}} \ddot{\boldsymbol{\vartheta}} + \mathcal{C}_{\mathfrak{fl}}$$

$$\mathcal{M}_{\mathfrak{fl}} \triangleq H_{\mathfrak{fl}} \Phi_{\mathfrak{fl}} M_{\mathfrak{fl}} \Phi^*_{\mathfrak{fl}} H^*_{\mathfrak{fl}}$$

Mass matrix

$$\mathcal{C}_{\mathfrak{fl}} \triangleq H_{\mathfrak{fl}} \Phi_{\mathfrak{fl}} (M_{\mathfrak{fl}} \Phi^*_{\mathfrak{fl}} \mathfrak{a}_{\mathfrak{fl}} + \mathfrak{b}_{\mathfrak{fl}} + \mathfrak{K}\vartheta)$$

SKO model for tree flexible multibody systems

- We have satisfied all the requirement for an SKO model
  - Tree structure
  - SKO and SPO operator
  - Remaining spatial operators and operator forms of the equations of motion
  - Operator expressions for the mass matrix and Coriolis terms
- All of the analysis and algorithms for SKO models carry over to flexible body systems

Recursive inverse dynamics

$$
\begin{cases}
\mathcal{V}_{\mathfrak{fl}}(\mathfrak{n}+1) = 0, \quad \alpha_{\mathfrak{fl}}(\mathfrak{n}+1) = \mathbf{0} \\
\textbf{for } \boldsymbol{k} \quad \mathfrak{n}\cdots 1 \\
\quad \mathcal{V}_{\mathfrak{fl}}(\mathrm{k}) = \Phi^*_{\mathfrak{fl}}(\mathrm{k}+1,\mathrm{k})\mathcal{V}_{\mathfrak{fl}}(\mathrm{k}+1) + \mathrm{H}^*_{\mathfrak{fl}}(\mathrm{k})\dot{\boldsymbol{\vartheta}}(\mathrm{k}) \\
\quad \alpha_{\mathfrak{fl}}(\mathrm{k}) = \Phi^*_{\mathfrak{fl}}(\mathrm{k}+1,\mathrm{k})\alpha_{\mathfrak{fl}}(\mathrm{k}+1) + \mathrm{H}^*_{\mathfrak{fl}}(\mathrm{k})\ddot{\boldsymbol{\vartheta}}(\mathrm{k}) + \mathfrak{a}_{\mathfrak{fl}}(\mathrm{k}) \\
\textbf{end loop}
\end{cases}
$$

scatter recursions

$$
\begin{cases}
\mathfrak{f}_{\mathfrak{fl}}(0) = \mathbf{0} \\
\textbf{for } \boldsymbol{k} \quad 1\cdots \mathfrak{n} \\
\quad \mathfrak{f}_{\mathfrak{fl}}(\mathrm{k}) = \Phi_{\mathfrak{fl}}(\mathrm{k},\mathrm{k}-1)\mathfrak{f}_{\mathfrak{fl}}(\mathrm{k}-1) + \mathrm{M}_{\mathfrak{fl}}(\mathrm{k})\alpha_{\mathfrak{fl}}(\mathrm{k}) + \mathfrak{b}_{\mathfrak{fl}}(\mathrm{k}) + \mathfrak{K}(\mathrm{k})\vartheta(\mathrm{k}) \\
\quad \mathcal{T}_{\mathfrak{fl}}(\mathrm{k}) = \mathrm{H}_{\mathfrak{fl}}(\mathrm{k})\mathfrak{f}_{\mathfrak{fl}}(\mathrm{k}) \\
\textbf{end loop}
\end{cases}
$$

gather recursions

ATBI Expressions & Analysis

Flexible body ATBI recursion

$$
\begin{cases}
\mathcal{P}^+_{fl}(0) = 0 \\
\textbf{for } \boldsymbol{k} \quad \mathbf{1 \cdots n} \\
\mathcal{P}_{fl}(k) = \Phi_{fl}(k, k-1)\mathcal{P}^+_{fl}(k-1)\Phi^*_{fl}(k, k-1) + \mathrm{M}_{fl}(k) \\
\mathcal{D}_{fl}(k) = \mathrm{H}_{fl}(k)\mathcal{P}_{fl}(k)\mathrm{H}^*_{fl}(k) \\
\mathcal{G}_{fl}(k) = \mathcal{P}_{fl}(k)\mathrm{H}^*_{fl}(k)\mathcal{D}^{-1}_{fl}(k) \\
\mathcal{K}_{fl}(k+1, k) = \Phi_{fl}(k+1, k)\mathcal{G}_{fl}(k) \\
\overline{\tau}(k) = \mathbf{I} - \mathcal{G}_{fl}(k)\mathrm{H}_{fl}(k) \\
\mathcal{P}^+_{fl}(k) = \overline{\tau}(k)\mathcal{P}_{fl}(k) \\
\Psi_{fl}(k+1, k) = \Phi_{fl}(k+1, k)\overline{\tau}(k) \\
\textbf{end loop}
\end{cases}
$$

ATBI operators

$$\mathcal{D}_{fl} \triangleq H_{fl}\mathcal{P}_{fl}H^*_{fl} = \text{diag}\left\{\mathcal{D}_{fl}(k)\right\}_{k=1}^{n} \quad \in \mathcal{R}^{\mathcal{N}_{fl}\times\mathcal{N}_{fl}}$$

$$\mathcal{G}_{fl} \triangleq \mathcal{P}_{fl}H^*_{fl}\mathcal{D}_{fl}{}^{-1} = \text{diag}\left\{\mathcal{G}_{fl}(k)\right\}_{k=1}^{n} \quad \in \mathcal{R}^{\check{\mathcal{N}}\times\mathcal{N}_{fl}}$$

$$\mathcal{K}_{fl} \triangleq \mathcal{E}_{\Phi_{fl}}\mathcal{G}_{fl} \quad \in \mathcal{R}^{\check{\mathcal{N}}\times\mathcal{N}_{fl}}$$

$$\overline{\tau} \triangleq \mathbf{I} - \mathcal{G}_{fl}H_{fl} = \text{diag}\left\{\overline{\tau}(k)\right\}_{k=1}^{n} \quad \in \mathcal{R}^{\check{\mathcal{N}}\times\check{\mathcal{N}}}$$

$$\mathcal{E}_{\Psi_{fl}} \triangleq \mathcal{E}_{\Phi_{fl}}\overline{\tau} \quad \in \mathcal{R}^{\check{\mathcal{N}}\times\check{\mathcal{N}}}$$

ATBI SPO operator

$$\Psi_{\mathfrak{fl}} \triangleq (\mathbf{I} - \mathcal{E}_{\Psi_{\mathfrak{fl}}})^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \Psi_{\mathfrak{fl}}(2,1) & \mathbf{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{\mathfrak{fl}}(\mathfrak{n},1) & \Psi_{\mathfrak{fl}}(\mathfrak{n},2) & \dots & \mathbf{I} \end{pmatrix} \in \mathcal{R}^{\check{\mathcal{N}} \times \check{\mathcal{N}}}$$

Mass matrix inversion

$$\mathcal{M}_{\mathfrak{fl}} = [\mathbf{I} + \mathrm{H}_{\mathfrak{fl}}\Phi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]\mathcal{D}_{\mathfrak{fl}}[\mathbf{I} + \mathrm{H}_{\mathfrak{fl}}\Phi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]^*$$

$$[\mathbf{I} + \mathrm{H}_{\mathfrak{fl}}\Phi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]^{-1} = [\mathbf{I} - \mathrm{H}_{\mathfrak{fl}}\Psi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]$$

$$\mathcal{M}^{-1} = [\mathbf{I} - \mathrm{H}_{\mathfrak{fl}}\Psi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]^*\mathcal{D}_{\mathfrak{fl}}^{-1}[\mathbf{I} - \mathrm{H}_{\mathfrak{fl}}\Psi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]$$

$$\ddot{\vartheta} = [\mathbf{I} - \mathrm{H}_{\mathfrak{fl}}\Psi_{\mathfrak{fl}}\mathcal{K}_{\mathfrak{fl}}]^*\mathcal{D}_{\mathfrak{fl}}^{-1}\Big[\mathcal{T}_{\mathfrak{fl}} - \mathrm{H}_{\mathfrak{fl}}\Psi_{\mathfrak{fl}}\{\mathcal{K}_{\mathfrak{fl}}\mathcal{T}_{\mathfrak{fl}} + \mathcal{P}_{\mathfrak{fl}}\mathfrak{a}_{\mathfrak{fl}} + \mathfrak{b}_{\mathfrak{fl}} + \mathfrak{K}\vartheta\}\Big] - \mathcal{K}_{\mathfrak{fl}}^*\Psi_{\mathfrak{fl}}^*\mathfrak{a}_{\mathfrak{fl}}$$

ATBI forward dynamics

$$
\begin{cases}
\mathfrak{z}^+(0) = \mathbf{0} \\
\textbf{for } \boldsymbol{k} \quad \mathbf{1 \cdots n} \\
\quad \mathfrak{z}(k) = \Phi_{fl}(k, k-1)\mathfrak{z}^+(k-1) + \mathcal{P}_{fl}(k)\mathfrak{a}_{fl}(k) + \mathfrak{b}_{fl}(k) + \mathfrak{K}(k)\vartheta(k) \\
\quad \epsilon(k) = \mathcal{T}_{fl}(k) - H_{fl}(k)\mathfrak{z}(k) \\
\quad \nu(k) = \mathcal{D}_{fl}^{-1}(k)\epsilon(k) \\
\quad \mathfrak{z}^+(k) = \mathfrak{z}(k) + \mathcal{G}_{fl}(k)\epsilon(k) \\
\textbf{end loop}
\end{cases}
$$

gather recursion

$$
\begin{cases}
\alpha_{fl}(n+1) = \mathbf{0} \\
\textbf{for } \boldsymbol{k} \quad \mathbf{n \cdots 1} \\
\quad \alpha_{fl}^+(k) = \Phi_{fl}^*(k+1, k)\alpha_{fl}(k+1) \\
\quad \ddot{\vartheta}(k) = \nu(k) - \mathcal{G}_{fl}{}^*(k)\alpha_{fl}^+(k) \\
\quad \alpha_{fl}(k) = \alpha_{fl}^+(k) + H_{fl}^*(k)\ddot{\vartheta}(k) + \mathfrak{a}_{fl}(k) \\
\textbf{end loop}
\end{cases}
$$

scatter recursion

Optimization

- All of the SKO model algorithms apply directly
- However there is further optimization possible based on the sparsity of

$$\Phi_{fl}(k+1,k) \triangleq \begin{pmatrix} \mathbf{0} & \Pi^*(\mathbb{O}^+_{k+1})\phi(\mathbb{O}^+_k,k) \\ \mathbf{0} & \phi(k+1,k) \end{pmatrix}$$

$$H_{fl}(k) \triangleq \begin{pmatrix} \mathbf{I} & -\Pi^*_{\mathbb{B}}(\mathbb{O}_k) \\ \mathbf{0} & H_{\mathbb{B}}(k) \end{pmatrix}$$

- Furthermore, modal integrals can be used to simplify the evaluation of the gyroscopic terms

Recap

Summary

- Introduced flexible bodies
- Showed nodal formulation followed by modal formulation for a flexible body
- Developed equations of motion for a single body
- Developed recursive expressions for body velocities, leading to an SKO model for tree systems
- Summarized applicability of SKO model analysis and algorithms to flex body systems

SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

See https://dartslab.jpl.nasa.gov/References/index.php for publications and references on the SOA methodology.

SOA Generalization Track Topics

8. **Graph theory based structure** – BWA matrices, connection to multibody systems
9. **Multibody graph systems** – generalization to tree topology rigid body systems, SKO/SPO operators, gather/scatter algorithms
10. **Closed-chain dynamics (cut-joint)** – holonomic and non-holonomic constraints, cut-joint method, operational space inertias, projected dynamics
11. **Closed-chain dynamics (constraint embedding)** – Multibody topology transformation and decomposition, aggregation, geared systems, constraint embedding for graph transformation, minimal coordinate closed-chain dynamics
12. **Flexible body dynamics** – Extension to flexible bodies, modal representations, recursive flexible body dynamics