Dynamics and Real-Time Simulation (DARTS) Laboratory

Spatial Operator Algebra (SOA)

7. O(N) Recursive Forward Dynamics

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https://dartslab.jpl.nasa.gov/

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SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

See https://dartslab.jpl.nasa.gov/References/index.php for publications and references on the SOA methodology.

Recap

Recap

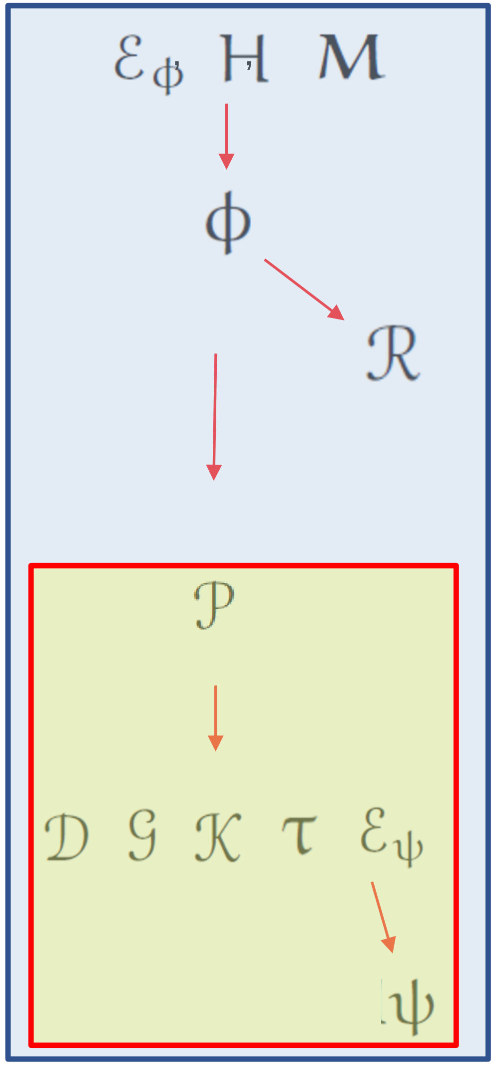
- Introduced ATBI spatial operators
- Developed several operator identities
- Developed Innovations Factorization of the mass matrix
  - Has square and invertible factors
  - Can reduce forward dynamics costs
- Developed expression for inverse of factors
- Developed operator expression for the mass matrix inverse

Spatial operators

- Velocity expression
- Jacobian
- Mass matrix NE factorization
- Lyapunov equation for CRBs
- Mass matrix decomposition
- Riccati equation for ATBI
- Several operator identities
- Mass matrix Innovations factorization
- Mass matrix determinant
- Mass matrix inverse and factorization

Have started to build up a vocabulary of spatial operators that can be used to express and manipulate the structure of dynamics quantities.

Now can see the rationale for the ***algebra*** *part of SOA from the analytical transformations and simplifications possible using the operators.*



spatial operators family

Recursive Computational Algorithms

- O(N) Gather and scatter recursions pattern
- O(N) Body velocities scatter recursion
- O(N) CRBs gather recursion
- $O(\mathcal{N}^2)$ mass matrix computation
- O(N) NE scatter/gather inverse dynamics
- $O(\mathcal{N}^2)$ inverse dynamics based mass matrix
- O(N) CRBs based inverse dynamics
- O(N) ATBI gather recursion
- $O(\mathcal{N}^2)$ forward dynamics

Can derive such low-cost scatter/gather algorithms usually by examination of the spatial operator expressions.

Forward Dynamics

System level equations of motion

System level equations of motion

$$\mathcal{V} = \phi^* H^* \dot{\theta}$$

$$\alpha = \phi^* (H^* \ddot{\theta} + \mathfrak{a})$$

$$\mathfrak{f} = \phi(\mathbf{M}\alpha + \mathfrak{b})$$

$$\mathcal{T} = H\mathfrak{f}$$

$$\mathcal{T} = H\,\phi\,\left[\mathbf{M}\,\phi^*\,\left(H^*\,\ddot{\theta} + \mathfrak{a}\right) + \mathfrak{b}\right] = \mathcal{M}(\theta)\ddot{\theta} + \mathcal{C}(\theta, \dot{\theta})$$

mass matrix $\mathcal{M}(\theta) = H\phi\mathbf{M}\phi^* H^* \in \mathcal{R}^{\mathcal{N}\times\mathcal{N}}$

Newton-Euler factorization

Coriolis terms $\mathcal{C}(\theta, \dot{\theta}) \triangleq H\phi(\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b}) \in \mathcal{R}^{\mathcal{N}}$

Inverse and Forward dynamics

Inverse dynamics:

- Given the state, and generalized accelerations, use the equations of motion to compute the generalized forces

$$\mathcal{T} = \mathcal{M}(\theta)\ddot{\theta} + \mathcal{C}(\theta, \dot{\theta})$$

- Important for feedforward control applications

Forward dynamics:

- Given the state, and generalized forces, solve the equations of motion to compute the generalized accelerations

$$\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$$

- Important for simulation applications

$\mathcal{O}(\mathcal{N}^3)$ forward dynamics

- Forward dynamics involves computing

$$\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$$

- The conventional approach would be to the compute the mass matrix, and to then solve the above matrix equation
- This process is of $\mathcal{O}(\mathcal{N}^3)$ computational complexity
- Can we do better?

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Spatial operator identities recap

$$\psi^{-1} - \phi^{-1} = \mathcal{K}\mathrm{H}$$

$$\begin{aligned}
\psi^{-1}\phi &= \mathbf{I} + \mathcal{K}\mathrm{H}\phi \\
\phi\psi^{-1} &= \mathbf{I} + \phi\mathcal{K}\mathrm{H} \\
\phi^{-1}\psi &= \mathbf{I} - \mathcal{K}\mathrm{H}\psi \\
\psi\phi^{-1} &= \mathbf{I} - \psi\mathcal{K}\mathrm{H}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi &= \mathrm{H}\psi \\
\phi\mathcal{K}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}] &= \psi\mathcal{K} \\
[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\mathrm{H}\psi &= \mathrm{H}\phi \\
\psi\mathcal{K}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}] &= \phi\mathcal{K}
\end{aligned}$$

These identities are very useful in transforming and simplifying operator expressions. We will see their use in a number of instances ahead.

Another operator identity

Earlier mass matrix expression

$$\mathcal{M}(\theta) \triangleq H\phi \mathbf{M} \phi^* H^*$$

dense

versus the similar expression

block-diagonal

$$H\psi \mathbf{M} \psi^* H^* = \mathcal{D}$$

Complex product of spatial operators collapses into just D!

The only difference is the use of ψ instead of ϕ !

$\mathcal{O}(\mathcal{N}^2)$ forward dynamics using Innovations Factorization

Innovations Factorization of the mass matrix

$$\mathcal{M} = [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\,\mathcal{D}\,[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^*$$

triangular with identity along block-diagonal — *Block-diagonal* — *Upper triangular with identity along block-diagonal*

- Forward dynamics involves computing $\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$
  With the NE factorization our options were limited to $\mathcal{O}(\mathcal{N}^3)$ complexity
- The new factors however can be computed at $\mathcal{O}(\mathcal{N}^2)$ cost
- Moreover these factors can be used to compute $\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$ at $\mathcal{O}(\mathcal{N}^2)$ cost as well.
- We have reduced the cost from $\mathcal{O}(\mathcal{N}^3)$ to $\mathcal{O}(\mathcal{N}^2)$ .This is progress – but can we do even better?

Mass Matrix Factorization & Inversion

$$\mathcal{M} = H\phi M\phi^* H^*$$

Analytical Newton-Euler factorization of the mass matrix

$$\mathcal{M} = [I + H\phi\mathcal{K}]\mathcal{D}[I + H\phi\mathcal{K}]^*$$

Analytical Innovations factorization of the mass matrix

$$[I + H\phi\mathcal{K}]^{-1} = I - H\psi\mathcal{K}$$

$$\mathcal{M}^{-1} = [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - H\psi\mathcal{K}]$$

Analytical operator expression for the mass matrix inverse

Forward dynamics expression

Analytical operator expression for the mass matrix inverse

$$\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$$

$$\mathcal{C}(\theta, \dot{\theta}) \triangleq H\phi(\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b}) \in \mathcal{R}^{\mathcal{N}}$$

$$\mathcal{M}^{-1} = [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - H\psi\mathcal{K}]$$

$$\ddot{\theta} \overset{7.20}{=} [\mathbf{I} - H\psi\mathcal{K}]^*\mathcal{D}^{-1}[\mathbf{I} - H\psi\mathcal{K}](\mathcal{T} - \mathcal{C})$$

$$\overset{5.25}{=} [\mathbf{I} - H\psi\mathcal{K}]^*\mathcal{D}^{-1}[\mathbf{I} - H\psi\mathcal{K}]\{\mathcal{T} - H\phi(\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b})\}$$

Complex - still have work to do!

Generalized Accelerations

Simplified expression for $\ddot{\theta}$

$$\ddot{\theta} \overset{7.20}{=} [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - H\psi\mathcal{K}](\mathcal{T} - \mathcal{C})$$

$$\overset{5.25}{=} [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - H\psi\mathcal{K}] \{\mathcal{T} - \boxed{H\phi(\mathbf{M}\phi^* \mathfrak{a} + \mathfrak{b})}\}$$

Coriolis terms

Claim:

$$\boxed{\ddot{\theta} = [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{D}^{-1} \left[\mathcal{T} - H\psi(\mathcal{K}\mathcal{T} + \boxed{\mathcal{P}\mathfrak{a} + \mathfrak{b})\right] - \mathcal{K}^* \psi^* \mathfrak{a}}}$$

Much simpler – has reduced products of 4 spatial operators down to just 2 such products!

Derivation of simpler $\ddot{\theta}$ expression

Step 1:

$$
\begin{aligned}
\ddot{\theta} &\overset{7.20}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}](\mathcal{T} - \mathcal{C}) \\
&\overset{5.25}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}] \{\mathcal{T} - \mathrm{H}\phi(\mathbf{M}\phi^* \mathfrak{a} + \mathfrak{b})\} \\
&= [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathcal{T} \\
&\quad - [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} \boxed{[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi(\mathbf{M}\phi^* \mathfrak{a} + \mathfrak{b})}
\end{aligned}
$$

Involves multiple operator products. Lets focus on simplifying this

Derivation of simpler $\ddot{\theta}$ expression (contd)

Step 2:

$$[\mathbf{I} - \mathcal{H}\psi\mathcal{K}]\mathcal{H}\phi = \mathcal{H}\psi$$

$$\phi\mathbf{M}\psi^* = \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*$$

$$[\mathbf{I} - \mathcal{H}\psi\mathcal{K}]\mathcal{H}\phi(\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b})$$
$$\overset{7.12}{=} \mathcal{H}\psi(\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b}) \overset{7.15}{=} \mathcal{H}\left(\left\{\psi\mathcal{P} + \mathcal{P}\tilde{\phi}^*\right\}\mathfrak{a} + \psi\mathfrak{b}\right)$$
$$\overset{7.3}{=} (\mathcal{H}\psi\mathcal{P} + \mathcal{D}\mathcal{K}^*\phi^*)\mathfrak{a} + \mathcal{H}\psi\mathfrak{b} = \mathcal{H}\psi(\mathcal{P}\mathfrak{a} + \mathfrak{b}) + \mathcal{D}\mathcal{K}^*\phi^*\mathfrak{a}$$

Much simpler – has reduced products of 3 spatial operators down to just solitary occurrences!

Derivation of simpler $\ddot{\theta}$ expression (contd)

Step 3:

$$
\begin{aligned}
&[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^*\mathcal{D}^{-1}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi(\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b}) \\
&\quad \overset{7.23}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^*\mathcal{D}^{-1}\left[\mathrm{H}\psi(\mathcal{P}\mathfrak{a} + \mathfrak{b}) + \mathcal{D}\mathcal{K}^*\phi^*\mathfrak{a}\right] \\
&\quad \overset{7.11}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^*\mathcal{D}^{-1}\mathrm{H}\psi(\mathcal{P}\mathfrak{a} + \mathfrak{b}) + \mathcal{K}^*\psi^*\mathfrak{a}
\end{aligned}
$$

$$\phi\mathcal{K}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}] = \psi\mathcal{K}$$

identity

Simplified expression for $\ddot{\theta}$

Operator manipulations helped simplify the generalized accelerations expression.

$$\ddot{\theta} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathcal{T} - [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi(\mathbf{M}\phi^* \mathfrak{a} + \mathfrak{b})$$

$$\ddot{\theta} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} \left[\mathcal{T} - \mathrm{H}\psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})\right] - \mathcal{K}^*\psi^*\mathfrak{a}$$

Reduced the number of multiple operator products from 4 to 2. ***Much simpler!***

Decomposing the Forward Dynamics Expression

Decomposing the $\ddot{\theta}$ expression

Breaking down the expression:

$$\ddot{\theta} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} \underbrace{[\mathcal{T} - \mathrm{H}\,\underbrace{\psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})}_{\mathfrak{z}}]}_{\epsilon}\ldots$$

$$\underbrace{\mathcal{D}^{-1}[\mathcal{T} - \mathrm{H}\psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})]}_{\nu} - \mathcal{K}^*\psi^*\mathfrak{a}$$

$$\mathfrak{z} \triangleq \psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})$$

$$\epsilon \triangleq \mathcal{T} - \mathrm{H}\mathfrak{z} \overset{7.25a}{=} \mathcal{T} - \mathrm{H}\psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})$$

$$\nu \triangleq \mathcal{D}^{-1}\epsilon \overset{7.25b}{=} \mathcal{D}^{-1}[\mathcal{T} - \mathrm{H}\psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})]$$

$$\ddot{\theta} \overset{7.21,7.25c}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \nu - \mathcal{K}^*\psi^*\mathfrak{a}$$

Alternative expression for $\mathfrak{z}$

Have $\mathfrak{z} \stackrel{\triangle}{=} \psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})$

Claim:

$$\mathfrak{z} = \phi\left[\mathcal{K}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}\right]$$

O(N) recursive gather algorithm

Derivation:

Have $\epsilon \stackrel{7.25b}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathcal{T} - \mathrm{H}\psi[\mathcal{P}\mathfrak{a} + \mathfrak{b}]$

Thus

$$\mathcal{T} \stackrel{G.7.1}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^{-1}\{\epsilon + \mathrm{H}\psi[\mathcal{P}\mathfrak{a} + \mathfrak{b}]\}$$

$$\stackrel{7.17,7.12}{=} [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\epsilon + \mathrm{H}\phi[\mathcal{P}\mathfrak{a} + \mathfrak{b}]$$

and

$$\mathfrak{z} \stackrel{7.25a}{=} \psi[\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b}] \stackrel{G.7.2}{=} \psi\mathcal{K}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\epsilon + \{\psi\mathcal{K}\mathrm{H}\phi + \psi\}[\mathcal{P}\mathfrak{a} + \mathfrak{b}]$$

$$\stackrel{7.10,7.12}{=} \phi\mathcal{K}\epsilon + \{\phi - \psi + \psi\}[\mathcal{P}\mathfrak{a} + \mathfrak{b}]$$

Defining and using $\mathfrak{z}^{+}$

Have $\mathfrak{z} = \phi\left[\mathcal{K}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}\right]$ Define $\mathfrak{z}^{+} \stackrel{\triangle}{=} \mathfrak{z} + \mathcal{G}\epsilon$

Claim:

$$\mathfrak{z} = \mathcal{E}_{\phi}\mathfrak{z}^{+} + \mathcal{P}\mathfrak{a} + \mathfrak{b}$$

Derivation:

Pre-multiplying and using $\phi^{-1} = (\mathbf{I} - \mathcal{E}_{\phi})$

we get

$$\mathfrak{z} = \mathcal{E}_{\phi}\mathfrak{z} + [\mathcal{K}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}] \stackrel{7.3}{=} \mathcal{E}_{\phi}(\mathfrak{z} + \mathcal{G}\epsilon) + \mathcal{P}\mathfrak{a} + \mathfrak{b} \stackrel{7.27}{=} \mathcal{E}_{\phi}\mathfrak{z}^{+} + \mathcal{P}\mathfrak{a} + \mathfrak{b}$$

Alternative expression for $\mathfrak{z}^+$

Have $\mathfrak{z}^+ \stackrel{\triangle}{=} \mathfrak{z} + \mathcal{G}\epsilon$

Claim:

$$\mathfrak{z}^+ \stackrel{\triangle}{=} \phi\left[\mathcal{G}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}\right]$$

Derivation:

$$\mathfrak{z}^+ \stackrel{7.27}{=} \mathfrak{z} + \mathcal{G}\epsilon \stackrel{7.26}{=} \phi[\mathcal{K}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}] + \mathcal{G}\epsilon$$
$$\stackrel{7.3}{=} \tilde{\phi}\mathcal{G}\epsilon + \phi[\mathcal{P}\mathfrak{a} + \mathfrak{b}] + \mathcal{G}\epsilon = \phi[\mathcal{G}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}]$$

Body accelerations

Body acceleration expression

Previously $\alpha = \phi^*(H^*\ddot{\theta} + \mathfrak{a})$

Claim:

$$\boxed{\alpha = \psi(H^*\nu + \mathfrak{a})}$$

O(N) recursive scatter algorithm

Derivation:

$$\ddot{\theta} \overset{7.21,7.25c}{=} [\mathbf{I} - H\psi\mathcal{K}]^*\nu - \mathcal{K}^*\psi^*\mathfrak{a}$$

$$\alpha \overset{5.23}{=} \phi^*[H^*\ddot{\theta} + \mathfrak{a}] \overset{7.25d}{=} \phi^*H^*\{[\mathbf{I} - H\psi\mathcal{K}]^*\nu - \mathcal{K}^*\psi^*\mathfrak{a}\} + \phi^*\mathfrak{a}$$

$$\overset{7.12}{=} \psi^*H^*\nu - \phi^*H^*\mathcal{K}^*\psi^*\mathfrak{a} + \phi^*\mathfrak{a} \overset{7.10}{=} \psi^*H^*\nu + \psi^*\mathfrak{a} = \psi^*[H^*\nu + \mathfrak{a}]$$

$$[\mathbf{I} - H\psi\mathcal{K}]H\phi = H\psi$$

$$\psi^{-1} - \phi^{-1} = \mathcal{K}H$$

Generalized acceleration expression

With

$$\alpha^{+} \stackrel{\triangle}{=} \operatorname{col}\left\{\alpha^{+}(k)\right\}_{k=1}^{n} \stackrel{6.9}{=} \mathcal{E}_{\phi}^{*}\alpha$$

$$\ddot{\theta} \stackrel{7.21,7.25c}{=} [\mathbf{I} - H\psi\mathcal{K}]^{*}\,\nu - \mathcal{K}^{*}\psi^{*}\mathfrak{a}$$

Claim:

$$\ddot{\theta} = \nu - \mathcal{K}^{*}\alpha = \nu - \mathcal{G}^{*}\alpha^{+}$$

Derivation:

$$\ddot{\theta} \stackrel{7.25d}{=} [\mathbf{I} - H\psi\mathcal{K}]^{*}\,\nu - \mathcal{K}^{*}\psi^{*}\mathfrak{a} = \nu - \mathcal{K}^{*}\psi^{*}H^{*}\nu - \mathcal{K}^{*}\psi^{*}\mathfrak{a} \stackrel{7.30}{=} \nu - \mathcal{K}^{*}\alpha$$

O(N) Recursive Forward Dynamics

Overall decomposed expressions

Putting it all together

$$
\begin{aligned}
\mathfrak{z} &\overset{7.28}{=} \mathcal{E}_{\phi}\mathfrak{z}^{+} + \mathcal{P}\mathfrak{a} + \mathfrak{b} \\
\mathfrak{z}^{+} &\overset{7.27}{=} \mathfrak{z} + \mathcal{G}\epsilon \\
\epsilon &\overset{7.25b}{=} \mathcal{T} - \mathrm{H}\mathfrak{z} \\
\nu &\overset{7.25c}{=} \mathcal{D}^{-1}\epsilon \\
\alpha^{+} &= \mathcal{E}_{\phi}^{*}\alpha \\
\ddot{\theta} &\overset{7.31}{=} \nu - \mathcal{G}^{*}\alpha^{+} \\
\alpha &= \alpha^{+} + \mathrm{H}^{*}\ddot{\theta} + \mathfrak{a}
\end{aligned}
$$

O(N) ATBI forward dynamics algorithm

ATBI recursion from before

$$\begin{cases} \mathcal{P}^+(0) = \mathbf{0}, \quad \mathfrak{z}^+(0) = \mathbf{0}, \quad \mathcal{T}(0) = \mathbf{0}, \quad \overline{\tau}(0) = \mathbf{0} \\ \textbf{for } k \quad 1 \cdots n \\ \quad \mathcal{P}(k) = \phi(k,k-1)\mathcal{P}^+(k-1)\phi^*(k,k-1) + M(k) \\ \quad \mathcal{D}(k) = H(k)\mathcal{P}(k)H^*(k) \\ \quad \mathcal{G}(k) = \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k) \\ \quad \overline{\tau}(k) = \mathbf{I} - \mathcal{G}(k)H(k) \\ \quad \mathcal{P}^+(k) = \overline{\tau}(k)\mathcal{P}(k) \\ \quad \mathfrak{z}(k) = \phi(k,k-1)\mathfrak{z}^+(k-1) + \mathcal{P}(k)\mathfrak{a}(k) + \mathfrak{b}(k) \\ \quad \epsilon(k) = \mathcal{T}(k) - H(k)\mathfrak{z}(k) \\ \quad \nu(k) = \mathcal{D}^{-1}(k)\epsilon(k) \\ \quad \mathfrak{z}^+(k) = \mathfrak{z}(k) + \mathcal{G}(k)\epsilon(k) \\ \textbf{end loop} \end{cases}$$

gather sweep

$$\begin{cases} \alpha(n+1) = \mathbf{0} \\ \textbf{for } k \quad n \cdots 1 \\ \quad \alpha^+(k) = \phi^*(k+1,k)\alpha(k+1) \\ \quad \ddot{\theta}(k) = \nu(k) - \mathcal{G}^*(k)\alpha^+(k) \\ \quad \alpha(k) = \alpha^+(k) + H^*(k)\ddot{\theta}(k) + \mathfrak{a}(k) \\ \textbf{end loop} \end{cases}$$

scatter sweep

O(N) computational complexity, fastest available algorithm

Structure of the ATBI O(N) recursions

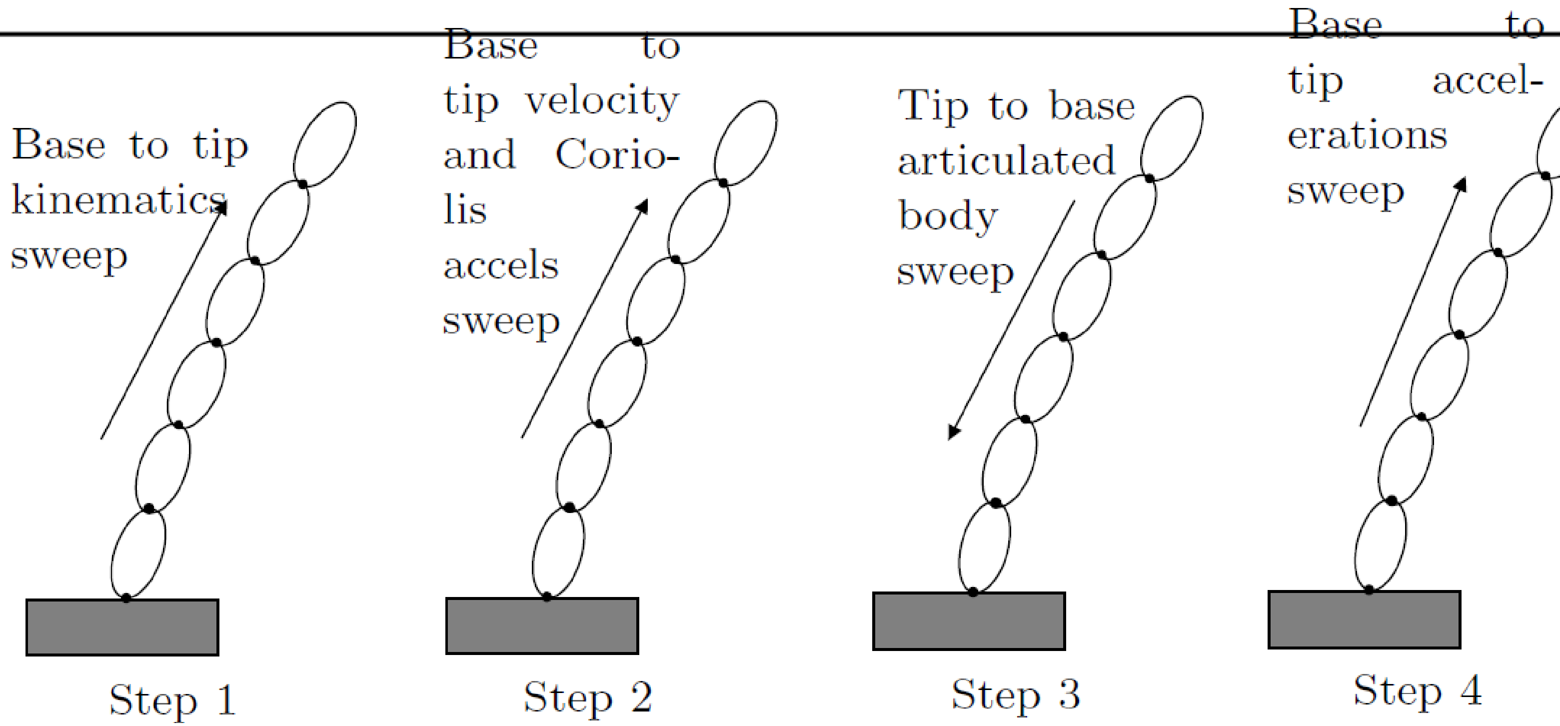
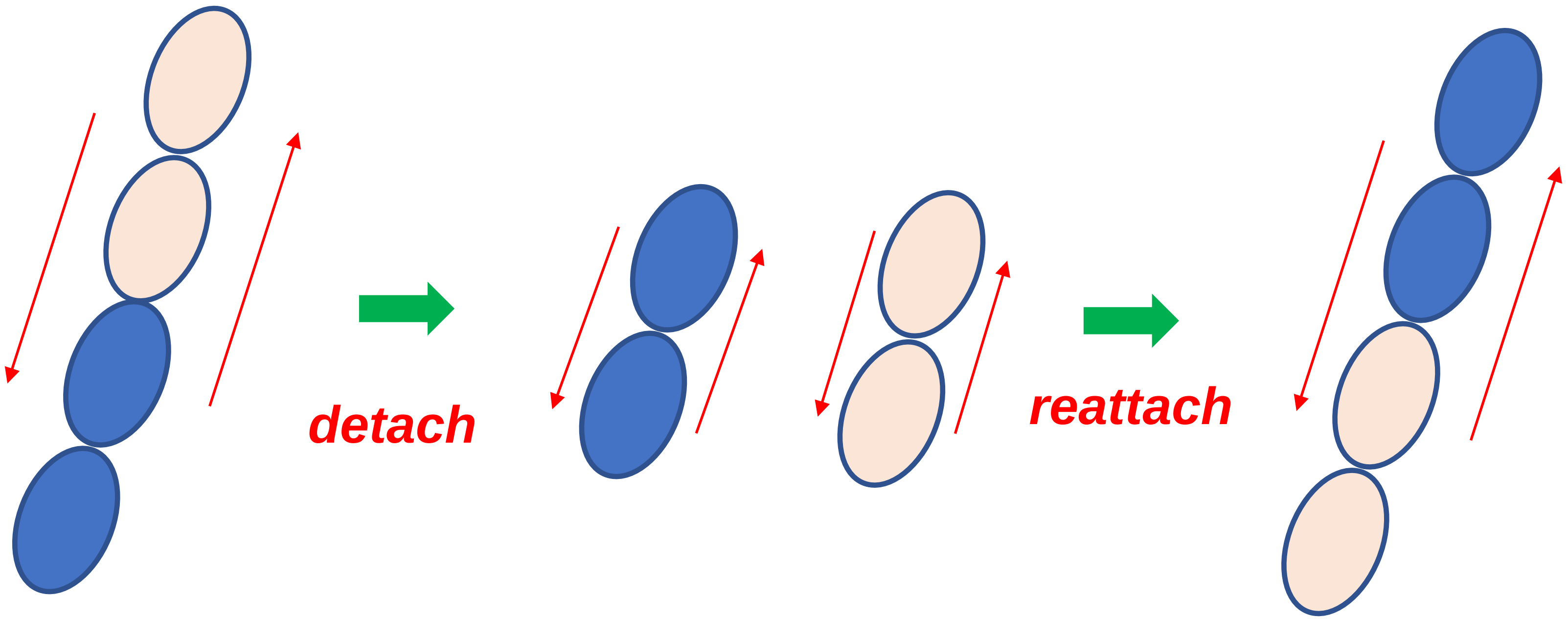


Fig. 7.1 Structure of the $\mathcal{O}(\mathcal{N})$ AB Algorithm 7.1 for serial-chain forward dynamics

Comments on the ATBI algorithm

- The ATBI algorithm cost scales linearly with the number of degrees of freedom, i.e. is O(N) – and is the fastest available to date
- This algorithm does not require the explicit computation of any of the operators
- It does not require the computation of the mass matrix inverse, or the mass matrix itself
- The algorithm applies to any size system
- The algorithm consists of gather & scatter sweeps, and these easily accommodate changes to the system from the addition or removal of bodies
- As we will see later, the ATBI algorithm continues to work for more general branched systems

Handling structural configuration changes



- *Multibody methods often have to work hard to handle structural changes – when they can*
- *The scatter and gather recursions however follow the instantaneous topological configuration (even addition and removal of bodies)*
- *Hence the SOA structure-based algorithms accommodate configuration changes automatically without missing a heartbeat!*

Alternative Expressions

Alternative expression for body accels

Claim:

$$\mathfrak{a} = \overline{\tau}^* \mathfrak{a}^+ + \mathrm{H}^* \mathrm{v} + \mathfrak{a}$$

$$\mathrm{v} = \mathcal{G}^* [\mathfrak{a} - \mathfrak{a}]$$

Derivation:

$$\begin{aligned}
\mathfrak{a} &\overset{7.30}{=} \psi^*[\mathrm{H}^* \mathrm{v} + \mathfrak{a}] \overset{7.8}{=} \tilde{\psi}^*[\mathrm{H}^* \mathrm{v} + \mathfrak{a}] + [\mathrm{H}^* \mathrm{v} + \mathfrak{a}] \\
&\overset{7.8}{=} \mathcal{E}_\psi^* \psi^*[\mathrm{H}^* \mathrm{v} + \mathfrak{a}] + [\mathrm{H}^* \mathrm{v} + \mathfrak{a}] \overset{7.25d}{=} \mathcal{E}_\psi^* \mathfrak{a} + \mathrm{H}^* \mathrm{v} + \mathfrak{a} \\
&\overset{7.32e}{=} \overline{\tau}^* \mathfrak{a}^+ + \mathrm{H}^* \mathrm{v} + \mathfrak{a}
\end{aligned}$$

The second equation is obtained by multiplying the first by $\mathcal{G}^*$

Inter-body forces

- In the absolute coordinate methods, the inter-body forces are the Lagrange multipliers associated with the constraints.
- These are not computed by the O(N) algorithm, but can be readily computed if needed.

Claim:

$$\mathfrak{f} = \mathcal{P}(\alpha - \mathfrak{a}) + \mathfrak{z} = \mathcal{P}^{+}\alpha^{+} + \mathfrak{z}^{+}$$

Derivation:

$$\begin{aligned}
\mathfrak{f} &\overset{5.23}{=} \phi[\mathbf{M}\alpha + \mathfrak{b}] \overset{7.30}{=} \phi\left[\mathbf{M}\psi^{*}(H^{*}\nu + \mathfrak{a}) + \mathfrak{b}\right] \\
&\overset{7.15}{=} [\tilde{\phi}\mathcal{P} + \mathcal{P}\psi](H^{*}\nu + \mathfrak{a}) + \phi\mathfrak{b} \overset{7.30}{=} \tilde{\phi}\mathcal{P}H^{*}\nu + \tilde{\phi}\mathcal{P}\mathfrak{a} + \mathcal{P}\alpha + \phi\mathfrak{b} \\
&\overset{7.3}{=} \phi\mathcal{K}\epsilon + \phi\mathcal{P}\mathfrak{a} - \mathcal{P}\mathfrak{a} + \mathcal{P}\alpha + \phi\mathfrak{b} \overset{7.26}{=} \mathcal{P}[\alpha - \mathfrak{a}] + \mathfrak{z}
\end{aligned}$$

$$\mathfrak{z} = \phi\left[\mathcal{K}\epsilon + \mathcal{P}\mathfrak{a} + \mathfrak{b}\right]$$

Similarly can show second equality.

Estimation theory parallels

$$
\begin{cases}
\mathcal{P}^+(0) = \mathbf{0}, \quad \mathfrak{z}^+(0) = \mathbf{0}, \quad \mathcal{T}(0) = \mathbf{0}, \quad \bar{\tau}(0) = \mathbf{0} \\
\textbf{for } \boldsymbol{k} \quad 1 \cdots n \\
\quad \mathcal{P}(k) = \phi(k, k-1)\mathcal{P}^+(k-1)\phi^*(k, k-1) + M(k) \\
\quad \mathcal{D}(k) = H(k)\mathcal{P}(k)H^*(k) \\
\quad \mathcal{G}(k) = \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k) \\
\quad \bar{\tau}(k) = \mathbf{I} - \mathcal{G}(k)H(k) \\
\quad \mathcal{P}^+(k) = \bar{\tau}(k)\mathcal{P}(k) \\
\quad \boxed{\mathfrak{z}(k)} = \phi(k, k-1)\mathfrak{z}^+(k-1) + \mathcal{P}(k)\mathfrak{a}(k) + \mathfrak{b}(k) \\
\quad \epsilon(k) = \mathcal{T}(k) - H(k)\mathfrak{z}(k) \\
\quad \nu(k) = \mathcal{D}^{-1}(k)\epsilon(k) \\
\quad \mathfrak{z}^+(k) = \mathfrak{z}(k) + \mathcal{G}(k)\epsilon(k) \\
\textbf{end loop}
\end{cases}
$$

$$
\mathfrak{f} = \mathcal{P}(\alpha - \mathfrak{a}) + \mathfrak{z} = \mathcal{P}^+\alpha^+ + \mathfrak{z}^+
$$

$$
\begin{cases}
\alpha(n+1) = \mathbf{0} \\
\textbf{for } \boldsymbol{k} \quad n \cdots 1 \\
\quad \alpha^+(k) = \phi^*(k+1, k)\alpha(k+1) \\
\quad \ddot{\theta}(k) = \nu(k) - \mathcal{G}^*(k)\alpha^+(k) \\
\quad \boxed{\alpha(k)} = \alpha^+(k) + H^*(k)\ddot{\theta}(k) + \mathfrak{a}(k) \\
\textbf{end loop}
\end{cases}
$$

causal filtering sweep

optimal filter estimate

optimal smoothed estimate

anti-causal co-state sweep

co-state

Including Gravity in the ATBI Forward Dynamics Algorithm

Including gravitational acceleration

The Coriolis term with gravity term is

$$\mathcal{C}(\theta, \dot{\theta}) = \mathrm{H}\phi\left[\mathbf{M}\phi^*(\mathfrak{a} + \boxed{\mathrm{E}^*\mathfrak{g}}) + \mathfrak{b}\right]$$

Claim:

$$\boxed{\ddot{\theta} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \left\{\mathcal{D}^{-1}\left[\mathcal{T} - \mathrm{H}\psi(\mathcal{K}\mathcal{T} + \mathcal{P}\mathfrak{a} + \mathfrak{b})\right] - \boxed{\mathcal{G}^*\bar{\mathrm{E}}^*\mathfrak{g}}\right\} - \mathcal{K}^*\psi^*\mathfrak{a}}$$

Derivation:

Define $\mathfrak{a}' \triangleq \mathfrak{a} + \mathrm{E}^*\mathfrak{g}$

And use in place of $\mathfrak{a}$ in the original proof of the forward dynamics expression and do further simplifications.

ATBI algorithm update

Define $\bar{\nu} \stackrel{\triangle}{=} \nu - \mathcal{G}^* \bar{E}^* \mathfrak{g}$

Claim:

$$\alpha = \psi^* \left(H^* \bar{\nu} + \mathfrak{a} \right)$$

$$\ddot{\theta} = [\mathbf{I} - H\psi\mathcal{K}]^* \bar{\nu} - \mathcal{K}^* \psi^* \mathfrak{a}$$

$$= \bar{\nu} - \mathcal{G}^* \alpha^+$$

Derivation: (first part)

$$\alpha \stackrel{5.23}{=} \phi^* [H^* \ddot{\theta} + \mathfrak{a}] \stackrel{7.30,7.36}{=} \psi^* [H^* \nu + \mathfrak{a}] - \phi^* H^* [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{G}^* \bar{E}^* \mathfrak{g}$$

$$\stackrel{7.12}{=} \psi^* [H^* \nu + \mathfrak{a}] - \psi^* H^* \mathcal{G}^* \bar{E}^* \mathfrak{g} = \psi^* \left[H^* \left(\nu - \mathcal{G}^* \bar{E}^* \mathfrak{g} \right) + \mathfrak{a} \right]$$

$$\stackrel{7.37}{=} \psi^* \left[H^* \bar{\nu} + \mathfrak{a} \right]$$

Replace ν with $\bar{\nu}$ in the original O(N) algorithm

Updated inter-body force computation

Earlier $\mathfrak{f} = \mathcal{P}(\alpha - \mathfrak{a}) + \mathfrak{z} = \mathcal{P}^{+}\alpha^{+} + \mathfrak{z}^{+}$

Claim:

$$\mathfrak{f} = \mathcal{P}\left(\alpha + \bar{\mathrm{E}}^{*}\mathfrak{g} - \mathfrak{a}\right) + \mathfrak{z} = \mathcal{P}^{+}\left(\alpha^{+} + \bar{\mathrm{E}}^{*}\mathfrak{g}\right) + \mathfrak{z}^{+}$$

Derivation: (first part)

$$\begin{aligned}
\mathfrak{f} &\overset{5.23}{=} \phi\left[\mathbf{M}(\alpha + \bar{\mathrm{E}}^{*}\mathfrak{g}) + \mathfrak{b}\right] \overset{7.38a}{=} \phi\left[\mathbf{M}\left[\psi^{*}(\mathrm{H}^{*}\bar{\nu} + \mathfrak{a}) + \bar{\mathrm{E}}^{*}\mathfrak{g}\right] + \mathfrak{b}\right] \\
&\overset{7.15}{=} \left[\tilde{\phi}\mathcal{P} + \mathcal{P}\psi\right](\mathrm{H}^{*}\bar{\nu} + \mathfrak{a}) + \phi\mathfrak{b} + \phi\mathbf{M}\bar{\mathrm{E}}^{*}\mathfrak{g} \\
&\overset{7.38a,7.37}{=} \tilde{\phi}\mathcal{P}\mathrm{H}^{*}\nu + \tilde{\phi}\mathcal{P}\mathfrak{a} + \mathcal{P}\alpha + \phi\mathfrak{b} + \left[\phi\mathbf{M} - \tilde{\phi}\mathcal{P}\mathrm{H}^{*}\mathcal{G}^{*}\right]\bar{\mathrm{E}}^{*}\mathfrak{g} \\
&\overset{5.41,7.25c}{=} \phi\mathcal{K}\epsilon + \phi\mathcal{P}\mathfrak{a} - \mathcal{P}\mathfrak{a} + \mathcal{P}\alpha + \phi\mathfrak{b} + \left[\phi\mathbf{M}\phi^{*} - \tilde{\phi}\mathcal{P}\mathrm{H}^{*}\mathcal{G}^{*}\phi^{*}\right]\mathrm{E}^{*}\mathfrak{g} \\
&\overset{7.26,7.16}{=} \mathcal{P}[\alpha - \mathfrak{a}] + \mathfrak{z} + \left[\tilde{\phi}\mathcal{P} + \mathcal{P}\phi^{*} - \tilde{\phi}\mathcal{P}\mathrm{H}^{*}\mathcal{G}^{*}\right]\mathrm{E}^{*}\mathfrak{g} \\
&\overset{5.41,7.3}{=} \mathcal{P}\left[\alpha + \bar{\mathrm{E}}^{*}\mathfrak{g} - \mathfrak{a}\right] + \mathfrak{z} + \left[\tilde{\phi}\mathcal{P} - \tilde{\phi}\mathcal{P}\mathrm{H}^{*}\mathcal{G}^{*}\right]\mathrm{E}^{*}\mathfrak{g} \\
&= \mathcal{P}\left[\alpha + \bar{\mathrm{E}}^{*}\mathfrak{g} - \mathfrak{a}\right] + \mathfrak{z} + \tilde{\phi}\mathcal{P}\bar{\tau}^{*}\mathrm{E}^{*}\mathfrak{g} = \mathcal{P}\left[\alpha + \bar{\mathrm{E}}^{*}\mathfrak{g} - \mathfrak{a}\right] + \mathfrak{z}
\end{aligned}$$

Including external forces in the ATBI Forward Dynamics Algorithm

Including external forces

The effect of external forces at nodes on the individual bodies can be accommodated by using the following modified step in the recursive algorithm:

$$\mathfrak{z}(k) = \phi(k, k-1)\mathfrak{z}^+(k-1) + \mathcal{P}(k)\mathfrak{a}(k) + \mathfrak{b}(k) - \sum_i \phi(B_k, O_k^i) f_{ext}^i(k)$$

external forces contribution for the kth body

Overall ATBI algorithm with gravity and external forces

$$
\begin{cases}
\mathcal{P}^+(0) = \mathbf{0}, \quad \mathfrak{z}^+(0) = \mathbf{0}, \quad \mathcal{T}(0) = \mathbf{0}, \quad \overline{\tau}(0) = \mathbf{0} \\
\textbf{for } k = 1 \cdots n \\
\quad \mathcal{P}(k) = \phi(k,k-1)\mathcal{P}^+(k-1)\phi^*(k,k-1) + M(k) \\
\quad \mathcal{D}(k) = H(k)\mathcal{P}(k)H^*(k) \\
\quad \mathcal{G}(k) = \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k) \\
\quad \overline{\tau}(k) = \mathbf{I} - \mathcal{G}(k)H(k) \\
\quad \mathcal{P}^+(k) = \overline{\tau}(k)\mathcal{P}(k) \\
\quad \mathfrak{z}(k) = \phi(k,k-1)\mathfrak{z}^+(k-1) + \mathcal{P}(k)\mathfrak{a}(k) + \mathfrak{b}(k) - \boxed{\sum_i \phi(B_k, O_k^i) \mathfrak{f}_{ext}^i(k)} \\
\quad \epsilon(k) = \mathcal{T}(k) - H(k)\mathfrak{z}(k) \\
\quad \nu(k) = \mathcal{D}^{-1}(k)\epsilon(k) \\
\quad \mathfrak{z}^+(k) = \mathfrak{z}(k) + \mathcal{G}(k)\epsilon(k) \\
\textbf{end loop}
\end{cases}
$$

external forces term

$$
\begin{cases}
\alpha(n+1) = \mathbf{0} \\
\textbf{for } k = n \cdots 1 \\
\quad \alpha^+(k) = \phi^*(k+1,k)\alpha(k+1) \\
\quad \bar{\nu}(k) = \nu(k) - \boxed{\mathcal{G}^*(k)\mathfrak{g}} \\
\quad \ddot{\theta}(k) = \bar{\nu}(k) - \mathcal{G}^*(k)\alpha^+(k) \\
\quad \alpha(k) = \alpha^+(k) + H^*(k)\ddot{\theta}(k) + \mathfrak{a}(k) \\
\textbf{end loop}
\end{cases}
$$

gravity term

Progression of mass matrix expressions and related algorithms

$\mathcal{M} = H\phi M\phi^* \boldsymbol{H}^*$	$\rightarrow$	$O(\mathcal{N})$ Newton-Euler Inverse Dynamics
		$O(\mathcal{N}^2)$ Composite Body Algorithm for $\mathcal{M}$
		$O(\mathcal{N}^3)$ Forward Dynamics
$\mathcal{M} = [I + H\phi K]D[I + H\phi K]^*$	$\rightarrow$	$O(\mathcal{N}^2)$ Forward Dynamics
$[I + H\phi K]^{-1} = [I - H\psi K]$		
$\mathcal{M}^{-1} = [I - H\psi K]^* D^{-1}[I - H\psi K]$	$\rightarrow$	$O(\mathcal{N})$ Articulated Body Forward Dynamics
		$O(\mathcal{N}^2)$ Computation of $\mathcal{M}^{-1}$

The progressive derivation of spatial operator expressions leads to family of algorithms and reduction in computational cost.

Minimal coordinates and the ATBI algorithm

- One of the *claims to fame* of the minimal coordinates approach is the remarkable existence of the O(N) ATBI algorithm – in contrast with the $O(\mathcal{N}^3)$ cost of conventional methods.
  - The ATBI algorithm can be derived directly – without the use of SOA operators.
  - First versions by Vereshchagin and Armstrong in 1970's, and more general versions by Featherstone and Rodriguez in 1980s
  - Several others were published soon after – and all shown to be essentially the same ATBI algorithm with different notation
- However, there is a lot more to minimal coordinate dynamics than just the ATBI algorithm
- Exploring the broader analytical structure of minimal coordinate dynamics, and developing a unifying approach to low-cost algorithms (including ATBI algorithm) is the goal of the broader SOA approach

Minimal coordinates and the ATBI algorithm (contd)

- Despite its computational benefits, the ATBI algorithm is not used as widely as it should be
- Without proper mathematical tools, generalizing the ATBI algorithms from scratch to apply to the broad class of multibody systems can be very daunting
  - Body flexibility
  - General branching
  - Closed chain and prescribed motion
  - Configuration changes
- The use of the ATBI algorithm has been confined by and large to rigid body serial chains for robotic arm applications
- SOA provides the mathematical machinery to address this issue, so the work is done once to learn the framework, and then generalizations and algorithms become easy

Prescribed Motion and Hybrid Dynamics

Inverse and Forward dynamics

Inverse dynamics:

- Given the state, and generalized accelerations, use the equations of motion to compute the generalized forces

$$\mathcal{T} = \mathcal{M}(\theta)\ddot{\theta} + \mathcal{C}(\theta, \dot{\theta})$$

- N gen. accels known, N gen. forces unknown

Forward dynamics:

- Given the state, and generalized forces, solve the equations of motion to compute the generalized accelerations

$$\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$$

- N gen. forces known, N gen. accels unknown

Hybrid dynamics

- Given the state, and generalized accelerations, and the equations of motion

$$\mathcal{T} = \mathcal{M}(\theta)\ddot{\theta} + \mathcal{C}(\theta, \dot{\theta})$$

- Assume mix of N gen. accels and gen. forces known, compute the complementary N gen accels and forces
- When the known are all gen. accels, we have inverse dynamics, and when the known are all gen. forces, we have forward dynamics
- Hybrid dynamics is a generalization that covers both forward and inverse dynamics
- The known gen. accels are also said to be undergoing **‘prescribed motion’** in the context of forward dynamics

Examples of Hybrid dynamics applications

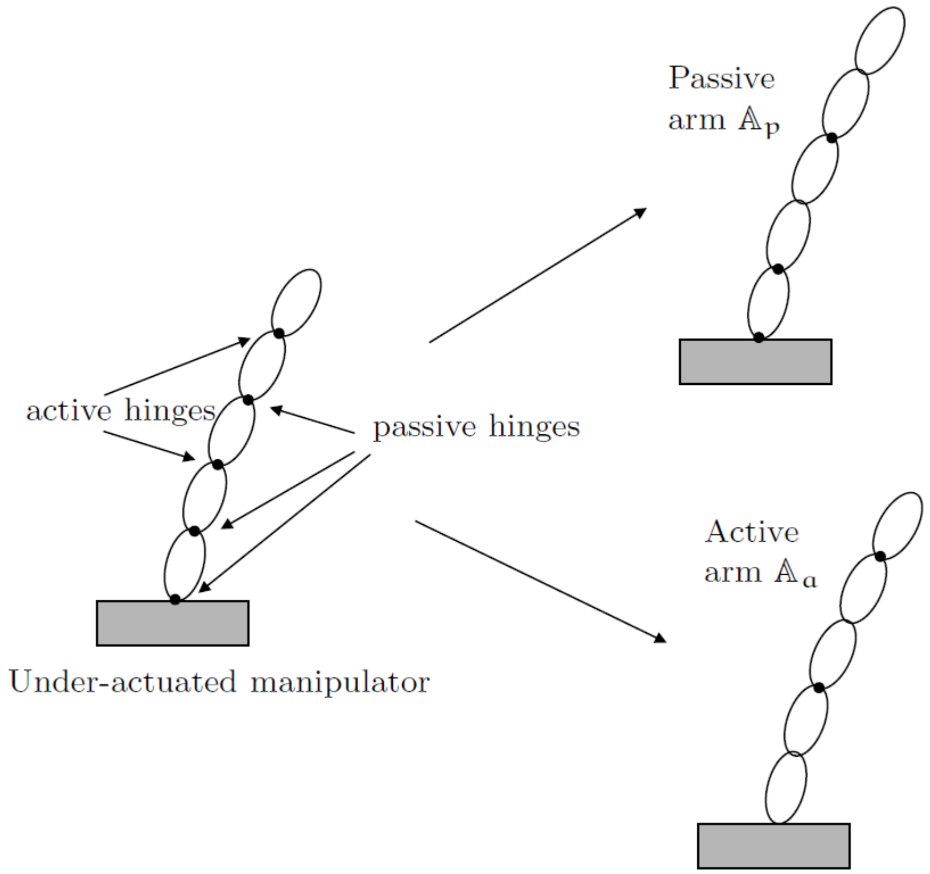
- Under-actuated systems control
  - Fewer actuators than dofs
  - Most mobile ground robots are under-actuated
- Docking, separation, jettison configuration change scenarios, eg.
  - Heat-shield separation
  - Multi-vehicle tandem operation
- Reduced order modeling
  - Freeze & thawing of dofs during run-time (molecular dynamics)
- Dofs with high-gain control
  - Eg. spacecraft actuator gimbals

Decomposition into active/passive dofs & systems

Refer to 'prescribed' dofs as 'active', and 'non-prescribed' as 'passive' dofs

Decompose the full (arm) system into **active** and **passive** systems

- Overall dofs same as original system



Partitioned equations of motion

With appropriate re-indexing, the equations of motion can be partitioned based on the active and passive dofs as follows

mass matrix of active system

known

$$\begin{pmatrix} \mathcal{M}_{aa} & \mathcal{M}_{ap} \\ \mathcal{M}^*_{ap} & \mathcal{M}_{pp} \end{pmatrix} \begin{bmatrix} \ddot{\theta}_a \\ \ddot{\theta}_p \end{bmatrix} + \begin{bmatrix} \mathcal{C}_a \\ \mathcal{C}_p \end{bmatrix} = \begin{bmatrix} \mathcal{T}_a \\ \mathcal{T}_p \end{bmatrix}$$

cross-coupling terms

mass matrix of passive system

unknown

$$\mathcal{M}_{ij} \triangleq H_i \phi \mathbf{M} \phi^* H_j^* \text{ and } \mathcal{C}_i \triangleq H_i \phi [\mathfrak{b} + \mathbf{M} \phi^* \mathfrak{a}]$$

Have known and unknown quantities on both sides of the equations of motion.

Rearranged equations of motion

We can rearrange the partitioned form so that all unknowns are on the left, and all knowns on the right as follows

$$\begin{bmatrix} \mathcal{T}_a \\ \ddot{\theta}_p \end{bmatrix} = \begin{pmatrix} \mathcal{S}_{aa} & \mathcal{S}_{ap} \\ -\mathcal{S}^*_{ap} & \mathcal{S}_{pp} \end{pmatrix} \begin{bmatrix} \ddot{\theta}_a \\ \mathcal{T}_p \end{bmatrix} + \begin{bmatrix} \mathcal{C}_a - \mathcal{S}_{ap}\mathcal{C}_p \\ -\mathcal{S}_{pp}\mathcal{C}_p \end{bmatrix}$$

unknown *known*

$$\mathcal{S}_{aa} \triangleq \mathcal{M}_{aa} - \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1}\mathcal{M}^*_{ap}$$

$$\mathcal{S}_{ap} \triangleq \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1}$$

$$\mathcal{S}_{pp} \triangleq \mathcal{M}_{pp}^{-1}$$

passive system mass matrix inverse

ATBI recursion for passive system

The ATBI quantities for the passive system can be computed as usual by a gather recursion

$$\mathcal{P}^+(0) = 0$$

for $\boldsymbol{k}$ $\quad \mathbf{1} \cdots \boldsymbol{n}$

$$\mathcal{P}(k) = \phi(k, k-1)\mathcal{P}^+(k-1)\phi^*(k, k-1) + M(k)$$

if $k \notin I_p$

$$\mathcal{P}^+(k) = \mathcal{P}(k)$$

skip active dof hinge

else

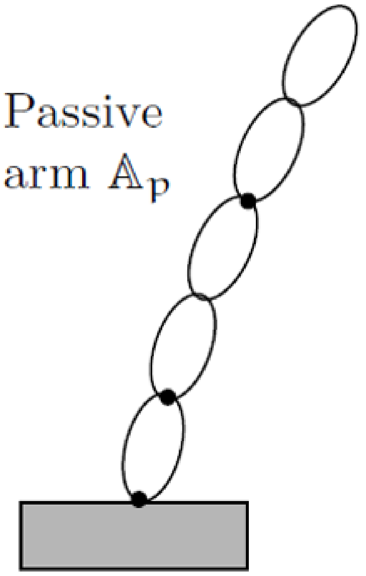
$$\mathcal{D}_p(k) = H_p(k)\mathcal{P}(k)H_p^*(k)$$
$$\mathcal{G}_p(k) = \mathcal{P}(k)H_p^*(k)\mathcal{D}_p^{-1}(k)$$
$$\mathcal{K}_p\overline{\tau}_p(k) = \mathbf{I} - \mathcal{G}_p(k)H_p(k)$$
$$\mathcal{P}^+(k) = \overline{\tau}_p(k)\mathcal{P}(k)$$

normal ATBI steps

end if

end loop

Passive arm $\mathbb{A}_p$



Passive system ATBI spatial operators

The usual ATBI spatial operators – except restricted to just the passive dofs.

$$
\begin{aligned}
\mathcal{D}_p &\triangleq H_p \mathcal{P} H_p^* \in \mathcal{R}^{\mathcal{N}_p \times \mathcal{N}_p} \\
\mathcal{G}_p &\triangleq \mathcal{P} H_p^* \mathcal{D}_p^{-1} \in \mathcal{R}^{6n \times \mathcal{N}_p} \\
\mathcal{K}_p &\triangleq \mathcal{E}_\phi \mathcal{G}_p \in \mathcal{R}^{6n \times \mathcal{N}_p} \\
\overline{\tau}_p &\triangleq \mathbf{I} - \mathcal{G}_p H_p \in \mathcal{R}^{6n \times 6n} \\
\mathcal{E}_\psi &\triangleq \mathcal{E}_\phi \overline{\tau}_p \in \mathcal{R}^{6n \times 6n}
\end{aligned}
$$

$$
\mathbf{M} = \mathcal{P} - \mathcal{E}_\psi \mathcal{P} \mathcal{E}_\psi^* = \mathcal{P} - \mathcal{E}_\phi \mathcal{P} \mathcal{E}_\psi^*
$$

passive system Riccati equation

Passive system mass matrix inversion

Since $\mathcal{M}_{pp}$ is a mass matrix, we can factorize and invert it using spatial operators as usual.

$$\mathcal{M}_{pp} = H_p \phi \mathbf{M} \phi^* H_p^*$$

NE factorization of the mass matrix

$$\mathcal{M}_{pp} = [\mathbf{I} + H_p \phi \mathcal{K}_p] \mathcal{D}_p [\mathbf{I} + H_p \phi \mathcal{K}_p]^*$$

Innovations factorization of the mass matrix

$$[\mathbf{I} + H_p \phi \mathcal{K}_p]^{-1} = [\mathbf{I} - H_p \psi \mathcal{K}_p]$$

$$\mathcal{M}_{pp}^{-1} = [\mathbf{I} - H_p \psi \mathcal{K}_p]^* \mathcal{D}_p^{-1} [\mathbf{I} - H_p \psi \mathcal{K}_p]$$

mass matrix inverse

Expression simplifications

Have

$$\begin{bmatrix} \mathcal{T}_a \\ \ddot{\theta}_p \end{bmatrix} = \begin{pmatrix} \mathcal{S}_{aa} & \mathcal{S}_{ap} \\ -\mathcal{S}_{ap}^* & \mathcal{S}_{pp} \end{pmatrix} \begin{bmatrix} \ddot{\theta}_a \\ \mathcal{T}_p \end{bmatrix} + \begin{bmatrix} \mathcal{C}_a - \mathcal{S}_{ap}\mathcal{C}_p \\ -\mathcal{S}_{pp}\mathcal{C}_p \end{bmatrix}$$

Claim:

$$\begin{aligned}
\mathcal{S}_{pp} &= [\mathbf{I} - H_p \psi \mathcal{K}_p]^* \mathcal{D}_p^{-1} [\mathbf{I} - H_p \psi \mathcal{K}_p] \\
\mathcal{S}_{ap} &= H_a \left\{ \psi \mathcal{K}_p + \mathcal{P} \psi^* H_p^* \mathcal{D}_p^{-1} [\mathbf{I} - H_p \psi \mathcal{K}_p] \right\} \\
&= H_a \left\{ (\psi - \mathcal{P}\Omega) \mathcal{K}_p + \mathcal{P} \psi^* H_p^* \mathcal{D}_p^{-1} \right\} \\
\mathcal{S}_{aa} &= H_a \left[\psi \mathbf{M} \psi^* - \mathcal{P} \Omega \mathcal{P} \right] H_a^* \\
&= H_a \left[(\psi - \mathcal{P}\Omega)\mathcal{P} + \mathcal{P} \tilde{\psi}^* \right] H_a^*
\end{aligned}$$

passive system mass matrix inverse

$$\Omega \triangleq \psi^* H_p^* \mathcal{D}_p^{-1} H_p \psi$$

operational space inertia (more later)

Derivation of $\mathcal{S}_{ap}$ expression

$$
\begin{aligned}
\mathcal{S}_{ap} &\overset{17.5}{=} \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1} \\
&\overset{17.3,17.11a}{=} H_a \phi \mathbf{M} \boxed{\phi^* H_p^* [\mathbf{I} - H_p \psi \mathcal{K}_p]^*} \mathcal{D}_p^{-1} [\mathbf{I} - H_p \psi \mathcal{K}_p] \\
&\overset{9.45}{=} H_a \boxed{\phi \mathbf{M} \psi^*} H_p^* \mathcal{D}_p^{-1} [\mathbf{I} - H_p \psi \mathcal{K}_p] \\
&\overset{9.41}{=} H_a \left\{ \phi \mathcal{K}_p + \mathcal{P} \psi^* H_p^* \mathcal{D}_p^{-1} \right\} [\mathbf{I} - H_p \psi \mathcal{K}_p] \\
&\overset{9.45}{=} H_a \left\{ \psi \mathcal{K}_p + \mathcal{P} \psi^* H_p^* \mathcal{D}_p^{-1} [\mathbf{I} - H_p \psi \mathcal{K}_p] \right\} \\
&\overset{17.12}{=} H_a \left\{ (\psi - \mathcal{P}\Omega) \mathcal{K}_p + \mathcal{P} \psi^* H_p^* \mathcal{D}_p^{-1} \right\}
\end{aligned}
$$

Derivation of $\mathcal{S}_{aa}$ expression

$$\mathcal{S}_{aa} \overset{17.5}{=} \mathcal{M}_{aa} - \mathcal{M}_{ap}\mathcal{M}_{pp}^{-1}\mathcal{M}_{ap}^{*} \overset{17.5}{=} \mathcal{M}_{aa} - \mathcal{S}_{ap}\mathcal{M}_{ap}^{*}$$

$$\overset{17.11b}{=} H_a\Big\{\phi - \boxed{\psi\mathcal{K}_p H_p \phi} - \mathcal{P}\psi^* H^* \mathcal{D}_p^{-1} * \boxed{[\mathbf{I} - H_p\psi\mathcal{K}_p]H_p\phi}\Big\}\mathbf{M}\phi^* H_a^*$$

$$\overset{9.43,9.45}{=} H_a\left\{\phi - (\phi - \psi) - \mathcal{P}\psi^* H^* \mathcal{D}_p^{-1} H_p \psi\right\}\mathbf{M}\phi^* H_a^*$$

$$= H_a\left\{\psi - \mathcal{P}\psi^* H^* \mathcal{D}_p^{-1} H_p \psi\right\}\mathbf{M}\phi^* H_a^*$$

$$\overset{17.12}{=} H_a\{\psi - \mathcal{P}\Omega\}\mathbf{M}\phi^* H_a^* \overset{10.38c}{=} H_a\left\{(\psi - \mathcal{P}\Omega)\mathcal{P} + \mathcal{P}\tilde{\psi}^*\right\}H_a^*$$

Hybrid Dynamics Solution

Operator expressions for solution

Have

$$\underline{\mathfrak{a}} \triangleq H_a^* \ddot{\theta}_a + \mathfrak{a}$$
$$\mathfrak{z} \triangleq \psi \left[\mathcal{K}_p \mathcal{T}_p + \mathfrak{b} + \mathcal{P}\underline{\mathfrak{a}}\right]$$
$$\epsilon_p \triangleq \mathcal{T}_p - H_p \mathfrak{z}$$
$$\nu_p \triangleq \mathcal{D}_p^{-1} \epsilon_p$$

Claim:

$$\alpha = \psi^* \left[H_p^* \nu_p + \underline{\mathfrak{a}}\right]$$
$$\ddot{\theta}_p = [\mathbf{I} - H_p \psi \mathcal{K}_p]^* \nu_p - \mathcal{K}_p^* \psi^* \underline{\mathfrak{a}} = \nu_p - \mathcal{K}_p{}^* \alpha$$
$$\mathcal{T}_a = H_a \left\{\mathfrak{z} + \mathcal{P}[\psi^* H_p^* \nu_p + \tilde{\psi}^* \underline{\mathfrak{a}}]\right\} = H_a \{\mathcal{P}\left[\alpha - \underline{\mathfrak{a}}\right] + \mathfrak{z}\}$$

Derivation of $\ddot{\theta}_p$ expression

Have
$$\begin{bmatrix} \mathcal{T}_a \\ \ddot{\theta}_p \end{bmatrix} = \begin{pmatrix} \mathcal{S}_{aa} & \mathcal{S}_{ap} \\ -\mathcal{S}^*_{ap} & \mathcal{S}_{pp} \end{pmatrix} \begin{bmatrix} \ddot{\theta}_a \\ \mathcal{T}_p \end{bmatrix} + \begin{bmatrix} \mathcal{C}_a - \mathcal{S}_{ap}\mathcal{C}_p \\ -\mathcal{S}_{pp}\mathcal{C}_p \end{bmatrix}$$

$$\boxed{\ddot{\theta}_p = \mathcal{S}_{pp}[\mathcal{T}_p - \mathcal{C}_p] - \mathcal{S}^*_{ap}\ddot{\theta}_a}$$

Thus
$$\begin{aligned}
\ddot{\theta}_p &= \mathcal{S}_{pp}[\mathcal{T}_p - \mathcal{C}_p] - \mathcal{S}^*_{ap}\ddot{\theta}_a \\
&\overset{9.45,17.11b}{=} [\mathbf{I} - H_p\psi\mathcal{K}_p]^*\mathcal{D}_p^{-1}\{\mathcal{T}_p - H_p\psi\,(\mathcal{K}_p\mathcal{T}_p + \mathfrak{b} + \mathbf{M}\phi^*\mathfrak{a})\} - \\
&\qquad \{\mathcal{K}_p{}^*\psi^* + [\mathbf{I} - H_p\psi\mathcal{K}_p]^*\mathcal{D}_p^{-1}H_p\psi\mathcal{P}\}\, H_a^*\ddot{\theta}_a \\
&\overset{9.45,9.41}{=} [\mathbf{I} - H_p\psi\mathcal{K}_p]^*\mathcal{D}_p^{-1}\{\mathcal{T}_p - H_p\psi\,(\mathcal{K}_p\mathcal{T}_p + \mathcal{P}\mathfrak{a} + \mathfrak{b})\} \\
&\qquad - \mathcal{K}_p^*\psi^*\mathfrak{a} - \{\mathcal{K}_p{}^*\psi^* + [\mathbf{I} - H_p\psi\mathcal{K}_p]^*\mathcal{D}_p^{-1}H_p\psi\mathcal{P}\}\, H_a^*\ddot{\theta}_a \\
&\overset{17.4}{=} [\mathbf{I} - H_p\psi\mathcal{K}_p]^*\mathcal{D}_p^{-1}\{\mathcal{T}_p - H_p\psi\,(\mathcal{K}_p\mathcal{T}_p + \mathcal{P}\underline{\mathfrak{a}} + \mathfrak{b})\} - \mathcal{K}_p^*\psi^*\underline{\mathfrak{a}} \\
&\overset{17.14}{=} [\mathbf{I} - H_p\psi\mathcal{K}_p]\nu_p - \mathcal{K}_p^*\psi^*\underline{\mathfrak{a}}
\end{aligned}$$

Derivation of body accels expression

$$\alpha \overset{9.1}{=} \phi^*[H^*\ddot{\theta} + \mathfrak{a}] \overset{17.1,17.4}{=} \phi^*[H_p^*\ddot{\theta}_p + \underline{\mathfrak{a}}]$$

$$\overset{17.15b}{=} \phi^* H_p^* \left\{[\mathbf{I} - H_p\psi\mathcal{K}_p]^*\nu_p - \mathcal{K}_p^*\psi^*\underline{\mathfrak{a}}\right\} + \phi^*\underline{\mathfrak{a}} \overset{9.43,9.45}{=} \psi^*[H_p^*\nu_p + \underline{\mathfrak{a}}]$$

$$\mathcal{S}_{ap}\mathcal{C}_p \overset{17.3,17.11b}{=} H_a\left\{\psi\mathcal{K}_p + \mathcal{P}\psi^*H_p^*\mathcal{D}_p^{-1}[\mathbf{I} - H_p\psi\mathcal{K}_p]\right\}H_p\phi[\mathfrak{b} + \mathbf{M}\phi^*\mathfrak{a}]$$

$$\overset{17.15b,9.45}{=} H_a\left\{(\phi - \psi) + \mathcal{P}\psi^*H_p^*\mathcal{D}_p^{-1}H_p\psi\right\}[\mathfrak{b} + \mathbf{M}\phi^*\mathfrak{a}]$$

$$\overset{13.2,17.12}{=} \mathcal{C}_a - H_a(\psi - \mathcal{P}\Omega)[\mathbf{M}\phi^*\mathfrak{a} + \mathfrak{b}]$$

$$= \mathcal{C}_a - H_a(\psi - \mathcal{P}\Omega)\mathbf{M}\phi^*\mathfrak{a} - H_a(\psi - \mathcal{P}\Omega)\mathfrak{b}$$

$$\overset{9.42,10.38a}{=} \mathcal{C}_a - H_a(\tilde{\psi}\mathcal{P} + \mathcal{P}\phi^* - \mathcal{P}(\phi^* - \psi^* + \Omega\mathcal{P}))\mathfrak{a} - H_a(\psi - \mathcal{P}\Omega)\mathfrak{b}$$

$$= \mathcal{C}_a - H_a(\psi - \mathcal{P}\Omega)[\mathcal{P}\mathfrak{a} + \mathfrak{b}] - H_a\mathcal{P}\tilde{\psi}^*\mathfrak{a}$$

Derivation of $\mathcal{T}_{\mathfrak{a}}$ expression

Have $$\mathcal{T}_{\mathfrak{a}} \overset{17.4}{=} \mathcal{S}_{\mathfrak{a}\mathfrak{a}}\ddot{\boldsymbol{\theta}}_{\mathfrak{a}} + \mathcal{S}_{\mathfrak{a}\mathfrak{p}}[\mathcal{T}_{\mathfrak{p}} - \mathcal{C}_{\mathfrak{p}}] + \mathcal{C}_{\mathfrak{a}}$$

$$\begin{aligned}
\mathcal{T}_{\mathfrak{a}} &\overset{17.4}{=} \mathcal{S}_{\mathfrak{a}\mathfrak{a}}\ddot{\boldsymbol{\theta}}_{\mathfrak{a}} + \mathcal{S}_{\mathfrak{a}\mathfrak{p}}[\mathcal{T}_{\mathfrak{p}} - \mathcal{C}_{\mathfrak{p}}] + \mathcal{C}_{\mathfrak{a}} \\
&\overset{17.11c,17.11b,17.16}{=} H_{\mathfrak{a}}\left\{(\psi - \mathcal{P}\Omega)\mathcal{P} + \mathcal{P}\tilde{\psi}^*\right\}H_{\mathfrak{a}}^*\ddot{\boldsymbol{\theta}}_{\mathfrak{a}} \\
&\qquad + H_{\mathfrak{a}}\left\{(\psi - \mathcal{P}\Omega)\mathcal{K}_{\mathfrak{p}} + \mathcal{P}\psi^* H_{\mathfrak{a}}^*\mathcal{D}_{\mathfrak{p}}^{-1}\right\}\mathcal{T}_{\mathfrak{p}} \\
&\qquad + H_{\mathfrak{a}}(\psi - \mathcal{P}\Omega)[\mathcal{P}\boldsymbol{\mathfrak{a}} + \boldsymbol{\mathfrak{b}}] + H_{\mathfrak{a}}\mathcal{P}\tilde{\psi}^*\boldsymbol{\mathfrak{a}} \\
&= H_{\mathfrak{a}}\Big[(\psi - \mathcal{P}\Omega)\left[\mathcal{P}H_{\mathfrak{a}}^*\ddot{\boldsymbol{\theta}}_{\mathfrak{a}} + \mathcal{K}_{\mathfrak{p}}\mathcal{T}_{\mathfrak{p}} + (\mathcal{P}\boldsymbol{\mathfrak{a}} + \boldsymbol{\mathfrak{b}})\right] \\
&\qquad + \mathcal{P}\left\{\tilde{\psi}^* H_{\mathfrak{a}}^*\ddot{\boldsymbol{\theta}}_{\mathfrak{a}} + \psi^* H_{\mathfrak{a}}^*\mathcal{D}_{\mathfrak{p}}^{-1}\mathcal{T}_{\mathfrak{p}} + \tilde{\psi}^*\boldsymbol{\mathfrak{a}}\right\}\Big] \\
&\overset{17.14,17.12,17.4}{=} H_{\mathfrak{a}}\left[\left[\mathbf{I} - \mathcal{P}\psi^* H_{\mathfrak{p}}^*\mathcal{D}_{\mathfrak{p}}^{-1}H_{\mathfrak{p}}\right]\mathfrak{z} + \mathcal{P}\left\{\tilde{\psi}^*\underline{\mathfrak{a}} + \psi^* H_{\mathfrak{p}}^*\mathcal{D}_{\mathfrak{p}}^{-1}\mathcal{T}_{\mathfrak{p}}\right\}\right] \\
&= H_{\mathfrak{a}}\left[\mathfrak{z} + \mathcal{P}\psi^* H_{\mathfrak{p}}^*\mathcal{D}_{\mathfrak{p}}^{-1}(\mathcal{T}_{\mathfrak{p}} - H_{\mathfrak{p}}\mathfrak{z}) + \mathcal{P}\tilde{\psi}^*\underline{\mathfrak{a}}\right] \\
&\overset{17.14}{=} H_{\mathfrak{a}}\left[\mathfrak{z} + \mathcal{P}\psi^* H_{\mathfrak{p}}^*\mathcal{D}_{\mathfrak{p}}^{-1}\epsilon + \mathcal{P}\tilde{\psi}^*\underline{\mathfrak{a}}\right] \\
&\overset{17.14}{=} H_{\mathfrak{a}}\left[\mathfrak{z} + \mathcal{P}\left(\psi^* H_{\mathfrak{p}}^*\nu + \tilde{\psi}^*\underline{\mathfrak{a}}\right)\right] \overset{17.15a}{=} H_{\mathfrak{a}}\left[\mathfrak{z} + \mathcal{P}\left(\alpha - \underline{\mathfrak{a}}\right)\right]
\end{aligned}$$

Hybrid Dynamics Algorithm

Recall solution operator expressions

$$\underline{\mathfrak{a}} \triangleq \boxed{H_{a}^{*}\ddot{\theta}_{a}} + \mathfrak{a}$$

active/prescribed gen accels term

$$\mathfrak{z} \triangleq \psi\left[\mathcal{K}_{p}\mathcal{T}_{p} + \mathfrak{b} + \mathcal{P}\underline{\mathfrak{a}}\right]$$

$$\epsilon_{p} \triangleq \mathcal{T}_{p} - H_{p}\mathfrak{z}$$

$$\nu_{p} \triangleq \mathcal{D}_{p}^{-1}\epsilon_{p}$$

$$\alpha = \psi^{*}\left[H_{p}^{*}\nu_{p} + \underline{\mathfrak{a}}\right]$$

$$\ddot{\theta}_{p} = \nu_{p} - \mathcal{K}_{p}{}^{*}\alpha$$

$$\mathcal{T}_{a} = H_{a}\{\mathcal{P}\left[\alpha - \underline{\mathfrak{a}}\right] + \mathfrak{z}\}$$

These expressions are very similar to regular ATBI forward dynamics and also map into similar O(N) ATBI gather/scatter recursions

O(N) Hybrid dynamics algorithm

Virtually the same structure as the regular O(N) ATBI algorithm

$$\begin{cases} \mathfrak{z}(0) = \mathbf{0} \\ \textbf{for } k \quad 1 \cdots n \\ \quad \begin{cases} \textbf{if } k \in I_a \\ \quad \mathfrak{z}(k) = \phi(k, k-1)\mathfrak{z}^+(k-1) + \mathfrak{b}(k) + \\ \qquad\qquad \mathcal{P}(k)[H^*(k)\ddot{\theta}_a(k) + \mathfrak{a}(k)] \\ \quad \mathfrak{z}^+(k) = \mathfrak{z}(k) \\ \textbf{else} \\ \quad \mathfrak{z}(k) = \phi(k, k-1)\mathfrak{z}^+(k-1) + \mathfrak{b}(k) + \mathcal{P}(k)\mathfrak{a}(k) \\ \quad \epsilon_p(k) = \mathcal{T}_p(k) - H(k)\mathfrak{z}(k) \\ \quad \mathfrak{z}^+(k) = \mathfrak{z}(k) + \mathcal{G}_p(k)\epsilon_p(k) \\ \quad \nu_p(k) = \mathcal{D}_p^{-1}\epsilon_p(k) \\ \textbf{end if} \end{cases} \\ \textbf{end loop} \end{cases}$$

$$\begin{cases} \alpha^+(n+1) = \mathbf{0} \\ \textbf{for } k \quad n \cdots 1 \\ \quad \alpha^+(k) = \phi^*(k+1, k)\alpha(k+1) \\ \qquad \begin{cases} \textbf{if } k \in I_a \\ \quad \mathfrak{f}(k) = \mathcal{P}(k)\alpha^+(k) + \mathfrak{z}(k) \\ \quad \mathcal{T}_a(k) = H(k)f(k) \\ \textbf{else} \\ \quad \ddot{\theta}_p(k) = \nu_p(k) - \mathcal{G}_p^*(k)\alpha^+(k) \\ \textbf{end if} \end{cases} \\ \quad \alpha(k) = \alpha^+(k) + H^*(k)\ddot{\theta}(k) + \mathfrak{a}(k) \\ \textbf{end loop} \end{cases}$$

altered steps for active/prescribed dofs

Recall: CRB-based Inverse dynamics algorithm

$$\mathfrak{f} = \mathcal{R}\alpha + \mathfrak{y} \qquad \mathfrak{y} \triangleq \phi\left[\mathfrak{b} + \mathcal{E}_{\phi}\mathcal{R}(H^*\ddot{\theta} + \mathfrak{a})\right]$$

- Use CRB gather algorithm to compute the CRB spatial inertias
- Compute the y values via a gather algorithm

$$\begin{cases} \mathfrak{y}^+(0) = \mathbf{0} \\ \textbf{for } k \quad 1\cdots n \\ \quad \mathfrak{y}(k) = \phi(k, k-1)\mathfrak{y}^+(k-1) + \mathfrak{b}(k) \\ \quad \mathfrak{y}^+(k) = \mathfrak{y}(k) + \mathcal{R}(k)\left[H^*(k)\ddot{\theta}(k) + \mathfrak{a}(k)\right] \\ \textbf{end loop} \end{cases}$$

The hybrid dynamics steps for the active dofs switch to these CRB inverse dynamics steps for active dofs!

- Compute the generalized forces

$$\mathcal{T}(k) \overset{5.21}{=} H(k)\mathfrak{f}(k) \overset{5.44}{=} H(k)\left[\mathcal{R}(k)\alpha(k) + \mathfrak{y}(k)\right]$$

Comments on O(N) hybrid dynamics algorithms

- Very similar in structure to the gather/scatter sweeps of the ATBI forward dynamics algorithm
- The computational cost scales linearly with the number of dofs
- The algorithm elegantly uses ATBI forward dynamics algorithm steps for passive dofs, and the CRB based inverse dynamics algorithm steps for the active dofs
- When all dofs are passive, we get the usual ATBI forward dynamics, and when all are active we get the CRB inverse dynamics
- Can flip the prescribed mode switch at run-time, and the algorithm automatically accommodates this

Example run-time change for prescribed motion

- Consider the separation or docking scenario between pair of vehicles
- Model the system as one system with a 6dof connection hinge
- When docked, set the hinge to prescribed motion with zero prescribed accelerations – essentially locks the hinge
- For undocking, disable the prescribed motion setting to allow the vehicles to move apart.

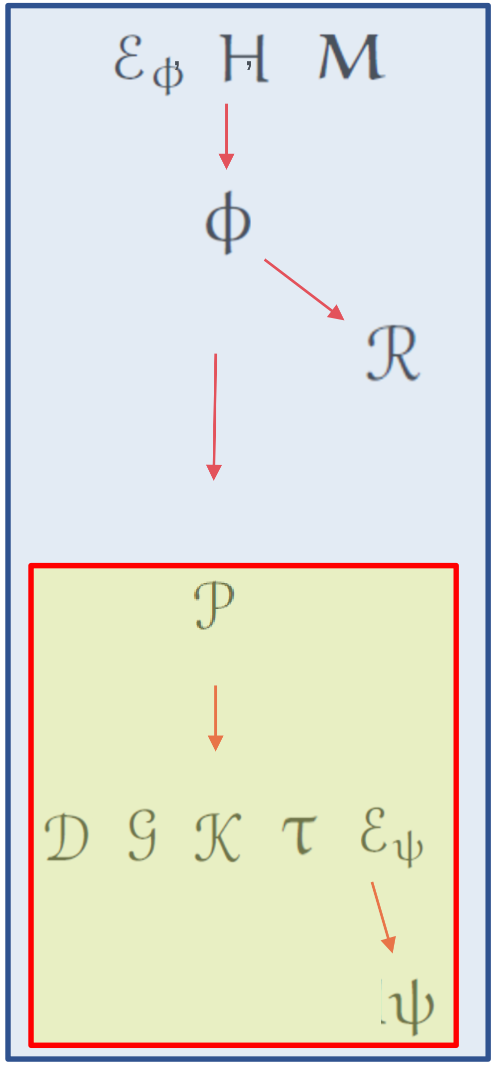
How far have we come?

Spatial operators (no new operators)

- Velocity expression
- Jacobian
- Mass matrix NE factorization
- Lyapunov equation for CRBs
- Mass matrix decomposition
- Riccati equation for ATBI
- Several operator identities
- Mass matrix Innovations factorization
- Mass matrix determinant
- Mass matrix inverse and factorization

Have started to build up a vocabulary of spatial operators that can be used to express and manipulate the structure of dynamics quantities.

Now can see the rationale for the ***algebra*** *part of SOA from the analytical transformations and simplifications possible using the operators.*



spatial operators family

Recursive Computational Algorithms

- O(N) Gather and scatter recursions pattern
- O(N) Body velocities scatter recursion
- O(N) CRBs gather recursion
- $O(N^2)$ mass matrix computation
- O(N) NE scatter/gather inverse dynamics
- $O(N^2)$ inverse dynamics based mass matrix
- O(N) CRBs based inverse dynamics
- O(N) ATBI gather recursion
- $O(N^2)$ forward dynamics
- O(N) ATBI forward dynamics
- O(N) hybrid dynamics

Can derive such low-cost scatter/gather algorithms usually by examination of the spatial operator expressions.

Summary

- Used the operator expression for the mass matrix inverse to develop the O(N) ATBI forward dynamics algorithm
- Described simple way to obtain inter-body forces if desired
- Developed extensions for handling gravity and external forces
- Developed O(N) generalized hybrid dynamics algorithm
  - Elegant combination of ATBI forward dynamics and CRB inverse dynamics

SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

SOA Generalization Track Topics

8. **Graph theory based structure** – BWA matrices, connection to multibody systems
9. **Tree topology systems** – generalization to tree topology rigid body systems, SKO/SPO operators, gather/scatter algorithms
10. **Closed-chain dynamics (cut-joint)** – holonomic and non-holonomic constraints, cut-joint method, operational space inertia, projected dynamics
11. **Closed-chain dynamics (constraint embedding)** – constraint embedding for graph transformation, minimal coordinate closed-chain dynamics
12. **Flexible body dynamics** – Extension to flexible bodies, modal representations, recursive flexible body dynamics