Dynamics and Real-Time Simulation (DARTS) Laboratory

Spatial Operator Algebra (SOA)

6. Mass Matrix Inverse

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https://dartslab.jpl.nasa.gov/

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SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

See https://dartslab.jpl.nasa.gov/References/index.php for publications and references on the SOA methodology.

Recap

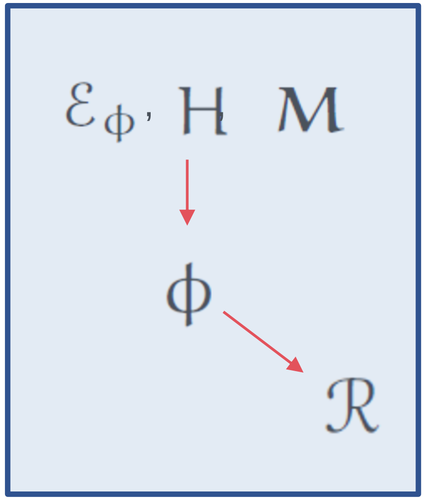
Recap

- Developed articulated body model for the decomposition of forces
  - Defined articulated body inertias and related quantities
  - Derived expression for residual forces
  - Developed O(N) gather algorithm for computing these quantities
- Described parallels with estimation theory

How far have we come?

Spatial operators

- Velocity expression
- Jacobian
- Mass matrix NE factorization
- Lyapunov equation for CRBs
- Mass matrix decomposition



spatial operators family

Have started to build up a vocabulary of spatial operators that can be used to express and manipulate the structure of dynamics quantities.

Recursive Computational Algorithms

- O(N) Gather and scatter recursions pattern
- O(N) Body velocities scatter recursion
- O(N) CRBs gather recursion
- $O(\mathcal{N}^2)$ mass matrix computation
- O(N) NE scatter/gather inverse dynamics
- $O(\mathcal{N}^2)$ inverse dynamics based mass matrix
- O(N) CRBs based inverse dynamics
- O(N) ATBI gather recursion

Can derive such low-cost scatter/gather algorithms usually by examination of the spatial operator expressions.

Articulated Body Inertia

Inter-body spatial force decompositions

- Force decompositions consist of inertia + residual terms
- From the equations of motion we had

$$\mathfrak{f}(k) = M(k)\alpha(k) + \phi(k, k-1)\mathfrak{f}(k-1)$$

*depends on **kth** body*

*depends on **all** bodies*

- Using CRBs we have

$$\mathfrak{f}(k) = \mathcal{R}(k)\alpha(k) + \mathfrak{y}(k)$$

*depends on **outboard** bodies only*

*depends on **outboard** generalized accels*

- Using ATBI we have

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)$$

*depends on **outboard** bodies only*

*depends on **outboard** generalized forces*

Defining $\psi(k+1,k)$

Defined the ***articulated body transformation matrix***

$$\psi(k+1,k) \triangleq \phi(k+1,k)\overline{\tau}(k)$$

- $\psi(k+1,k)$ is a 6x6 matrix like $\phi(k+1,k)$
- However it is typically singular
- It depends on hinge properties
- Unlike $\phi(k+1,k)$ which propagates across rigid bodies, $\psi(k+1,k)$ propagates across articulated bodies

Articulated body inertias algorithm

O(N) tip-to-base gather algorithm for ATBI quantities

$$\mathcal{P}^+(0) = \mathbf{0}, \quad \overline{\tau}(0) = \mathbf{0}$$

for $\boldsymbol{k}$ $\quad \mathbf{1 \cdots n}$

$$\psi(k, k-1) = \phi(k, k-1)\overline{\tau}(k-1)$$

$$\mathcal{P}(k) = \phi(k, k-1)\mathcal{P}^+(k-1)\phi^*(k, k-1) + M(k)$$

$$\mathcal{D}(k) = H(k)\mathcal{P}(k)H^*(k)$$

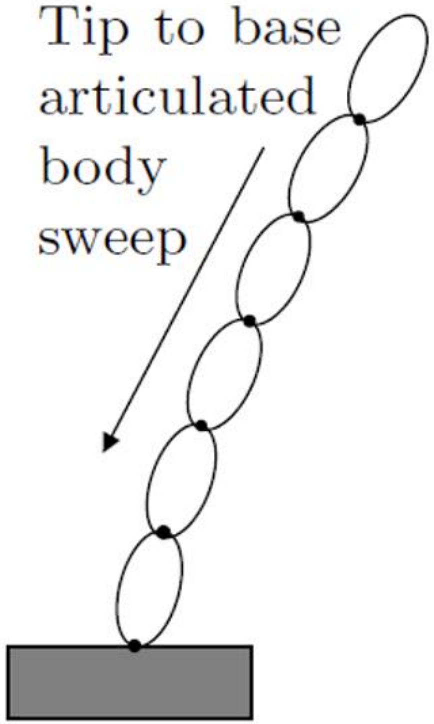
$$\mathcal{G}(k) = \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k)$$

$$\mathcal{K}(k+1, k) = \phi(k+1, k)\mathcal{G}(k)$$

$$\overline{\tau}(k) = \mathbf{I} - \mathcal{G}(k)H(k)$$

$$\mathcal{P}^+(k) = \overline{\tau}(k)\mathcal{P}(k)$$

end loop



Articulated Body Inertia Spatial Operators

ATBI spatial operators

Now define spatial operators using the ATBI quantities

$$\mathcal{P} \triangleq \text{diag}\left\{\mathcal{P}(k)\right\}_{k=1}^{n} \qquad \in \mathcal{R}^{6n\times 6n}$$

$$\mathcal{D} \triangleq \text{diag}\left\{\mathcal{D}(k)\right\}_{k=1}^{n} = H\mathcal{P}H^* \qquad \in \mathcal{R}^{\mathcal{N}\times\mathcal{N}}$$

$$\mathcal{G} \triangleq \text{diag}\left\{\mathcal{G}(k)\right\}_{k=1}^{n} = \mathcal{P}H^*\mathcal{D}^{-1} \qquad \in \mathcal{R}^{6n\times\mathcal{N}}$$

$$\mathcal{K} \triangleq \mathcal{E}_{\phi}\mathcal{G} \qquad \in \mathcal{R}^{6n\times\mathcal{N}}$$

$$\tau \triangleq \text{diag}\left\{\tau(k)\right\}_{k=1}^{n} = \mathcal{G}H \qquad \in \mathcal{R}^{6n\times 6n}$$

$$\overline{\tau} \triangleq \text{diag}\left\{\overline{\tau}(k)\right\}_{k=1}^{n} = \mathbf{I} - \tau \qquad \in \mathcal{R}^{6n\times 6n}$$

$$\mathcal{P}^+ \triangleq \text{diag}\left\{\mathcal{P}^+(k)\right\}_{k=1}^{n} = \overline{\tau}\mathcal{P}\overline{\tau}^* = \overline{\tau}\mathcal{P} = \mathcal{P}\overline{\tau}^* \qquad \in \mathcal{R}^{6n\times 6n}$$

$$\mathcal{E}_{\psi} \triangleq \mathcal{E}_{\phi}\overline{\tau} \qquad \in \mathcal{R}^{6n\times 6n}$$

Structure of the $\mathcal{K}$ spatial operator

$\mathcal{K}$ has the same structure as $\mathcal{E}_\phi$

$$\mathcal{K} \triangleq \mathcal{E}_\phi \mathcal{G}$$

$$\mathcal{K}(k+1,k) = \phi(k+1,k)\mathcal{G}(k)$$

block diagonal

$$\mathcal{K} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathcal{K}(2,1) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}(3,2) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathcal{K}(n,n-1) & \mathbf{0} \end{pmatrix}$$

Only non-zero entries are along the first sub-diagonal

Structure of $\mathcal{E}_\psi$

$\mathcal{E}_\psi$ also has the same structure as $\mathcal{E}_\phi$

$$\mathcal{E}_\psi \triangleq \mathcal{E}_\phi \overline{\tau}$$

$$\psi(k+1,k) \triangleq \phi(k+1,k)\overline{\tau}(k)$$

block diagonal

$$\mathcal{E}_\psi = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \psi(2,1) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \psi(3,2) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \psi(n,n-1) & \mathbf{0} \end{pmatrix}$$

$\mathcal{E}_\psi$ has same structure as $\mathcal{E}_\phi$

Like $\mathcal{E}_\phi$, $\mathcal{E}_\psi$ is nilpotent

$$\mathcal{E}_\psi = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \psi(2,1) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \psi(3,2) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \psi(n,n-1) & \mathbf{0} \end{pmatrix}$$

$$\mathcal{E}_\psi \triangleq \mathcal{E}_\phi \overline{\tau}$$

- Every power of $\mathcal{E}_\psi$ results in a matrix with the sub-diagonal shifted one step lower

$$\mathcal{E}_\mathbb{A} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \mathrm{X} & \cdot & \cdot & \cdot \\ \cdot & \mathrm{X} & \cdot & \cdot \\ \cdot & \cdot & \mathrm{X} & \cdot \end{pmatrix}, \; \mathcal{E}_\mathbb{A}^2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{X} & \cdot & \cdot & \cdot \\ \cdot & \mathrm{X} & \cdot & \cdot \end{pmatrix}, \; \mathcal{E}_\mathbb{A}^3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{X} & \cdot & \cdot & \cdot \end{pmatrix}$$

- At the nth power, the result is zero: $\mathcal{E}_\psi^n = \mathbf{0}$
- Hence $\mathcal{E}_\psi$ is **nilpotent**!

Structural properties of $\mathcal{E}_\psi$

- Strictly lower triangular, square, singular and nilpotent,
- Only the first sub-diagonal has nonzero elements
- The non-zero entries are the **configuration dependent** 6x6 inter-link articulated body transformation matrices (configuration dependent)

$$\mathcal{E}_\psi = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \psi(2,1) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \psi(3,2) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \psi(n,n-1) & \mathbf{0} \end{pmatrix}$$

Recall: Nilpotent matrices & inverses

- A square matrix $\mathfrak{U}$ is said to be nilpotent if one of its powers becomes 0, i.e. if for some n

$$\mathfrak{U}^n = \mathbf{0}$$

- For a nilpotent $\mathfrak{U}$, we have

$$(\mathbf{I} - \mathfrak{U})^{-1} = \mathbf{I} + \mathfrak{U} + \mathfrak{U}^2 + \cdots + \mathfrak{U}^{n-1}$$

1-resolvent

*Series expansion truncates after only a **finite** number of terms for nilpotent matrix, hence the 1-resolvent inverse is well defined*

ψ Is the 1-resolvent of $\mathcal{E}_\psi$

Analogous to

$$\phi \triangleq (\mathbf{I} - \mathcal{E}_\phi)^{-1} = \mathbf{I} + \mathcal{E}_\phi + \mathcal{E}_\phi^2 + \cdots + \mathcal{E}_\phi^{n-1}$$

$$\phi(i,j) = \phi(i,i-1) \cdots \phi(j+1,j)$$

define

$$\psi \triangleq (\mathbf{I} - \mathcal{E}_\psi)^{-1} = \mathbf{I} + \mathcal{E}_\psi + \mathcal{E}_\psi^2 + \cdots + \mathcal{E}_\psi^{n-1}$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \psi(2,1) & \mathbf{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \psi(n,1) & \psi(n,2) & \dots & \mathbf{I} \end{pmatrix}$$

$$\psi(i,j) \triangleq \psi(i,i-1) \cdots \psi(j+1,j) \quad \text{for} \quad i > j$$

$\tilde{\psi}$

Spatial operator

Analogous to

$$\tilde{\phi} \triangleq \phi - \mathbf{I}$$

Same as ϕ, except diagonal elements are now zero matrices

and

$$\tilde{\phi} = \mathcal{E}_{\phi}\phi = \phi\mathcal{E}_{\phi}$$

define

$$\tilde{\psi} \triangleq \psi\mathcal{E}_{\psi} = \mathcal{E}_{\psi}\psi = \psi - \mathbf{I}$$

Same as ψ, except diagonal elements are now zero matrices

Articulated vs Composite Body Models

Comparison of composite and articulated body models

Variable	Composite Body	Articulated Body
Relative hinge acceleration α $\alpha(k)$ α with respect to α^+	$\ddot{\theta}$ $\phi^* H^* \ddot{\theta}$ $\phi^*(k+1,k)\alpha(k+1) + H^*(k)\ddot{\theta}(k)$ $\alpha^+ + H^* \ddot{\theta}$	ν $\psi^* H^* \nu$ $\psi^*(k+1,k)\alpha(k+1) + H^*(k)\nu(k)$ $\overline{\tau}^* \alpha^+ + H^* \nu$ $(\overline{\tau}^* \alpha = \overline{\tau}^* \alpha^+)$
ν	$\mathcal{G}^* \alpha^+ + \ddot{\theta}$	$\mathcal{G}^* \alpha$
Effective inertia Relationship to $\mathbf{M}$	$\mathcal{R}$ $\mathcal{R} - \mathcal{E}_\phi \mathcal{R} \mathcal{E}_\phi^*$	$\mathcal{P}$ $\mathcal{P} - \mathcal{E}_\psi \mathcal{P} \mathcal{E}_\psi^*$
Inertia recursions	$\mathcal{R}^+(k) = \mathcal{R}(k)$ $\mathcal{R}(k+1) = \phi(k+1,k)\mathcal{R}^+(k)\phi^*(k+1,k) + M(k)$	$\mathcal{P}^+(k+1) = \overline{\tau}(k)\mathcal{P}(k)\overline{\tau}^*(k)$ $\mathcal{P}(k+1) = \phi(k+1,k)\mathcal{P}^+(k)\phi^*(k+1,k) + M(k)$
$\mathfrak{f}$ on (– side) Correction force (– side)	$\mathcal{R}\alpha + \mathfrak{y}$ $\mathfrak{y} = \tilde{\phi}\mathcal{R}H^* \ddot{\theta}$	$\mathcal{P}\alpha + \mathfrak{z}$ $\mathfrak{z} = \tilde{\phi}\mathcal{P}H^* \nu$
$\mathfrak{f}$ on + side Correction force (+ side)	$\mathcal{R}^+\alpha^+ + \mathfrak{y}^+$ $\mathfrak{y}^+ = \mathfrak{y} + \mathcal{R}H^* \ddot{\theta}$	$\mathcal{P}^+\alpha^+ + \mathfrak{z}^+$ $\mathfrak{z}^+ = \mathfrak{z} + \mathcal{P}H^* \nu$

Scatter Recursions for ψ same as for ϕ

Recall: Base-to-tips structure-based O(N) <u>scatter</u> recursion for ϕ

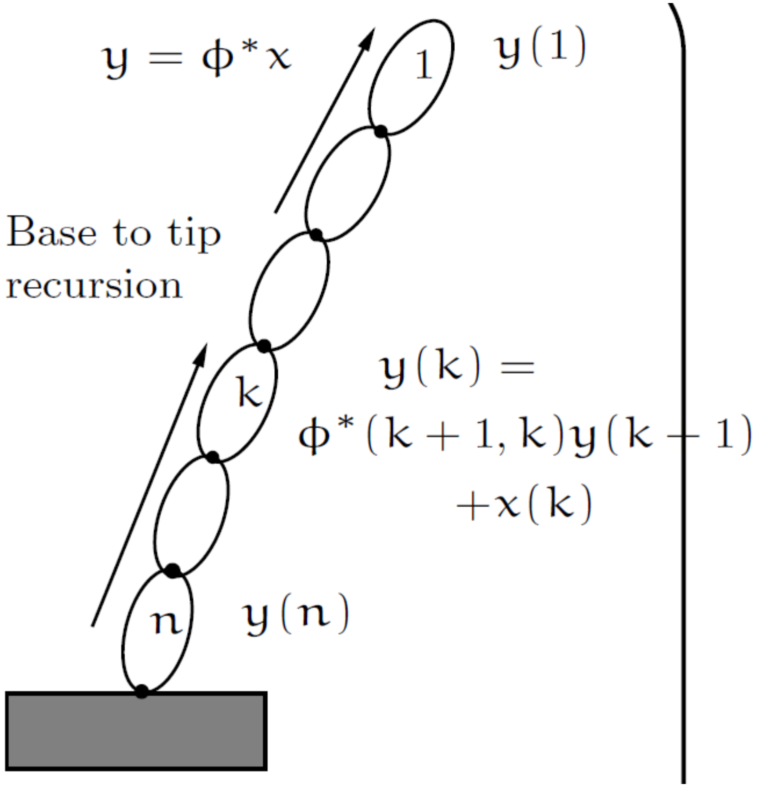
operator transpose/vector product

$$y = \phi^* x$$

- *Applies to any x*
- *Does not require explicit computation of ϕ at all*
- *Only depends on elements of $\mathcal{E}_\phi$*

$$\begin{cases} y(n+1) = 0 \\ \textbf{for } k \quad n \cdots 1 \\ \quad y(k) = \phi^*(k+1,k)y(k+1) + x(k) \\ \textbf{end loop} \end{cases}$$

O(N) structure-based, base-to-tip <u>scatter</u> recursion



Algorithm flow

Example – link velocity computation

Generalized base-to-tips structure-based scatter recursion

operator transpose/vector product

$$y = \mathbb{A}^* x$$

$$\mathbb{A} = \phi,\ \psi$$

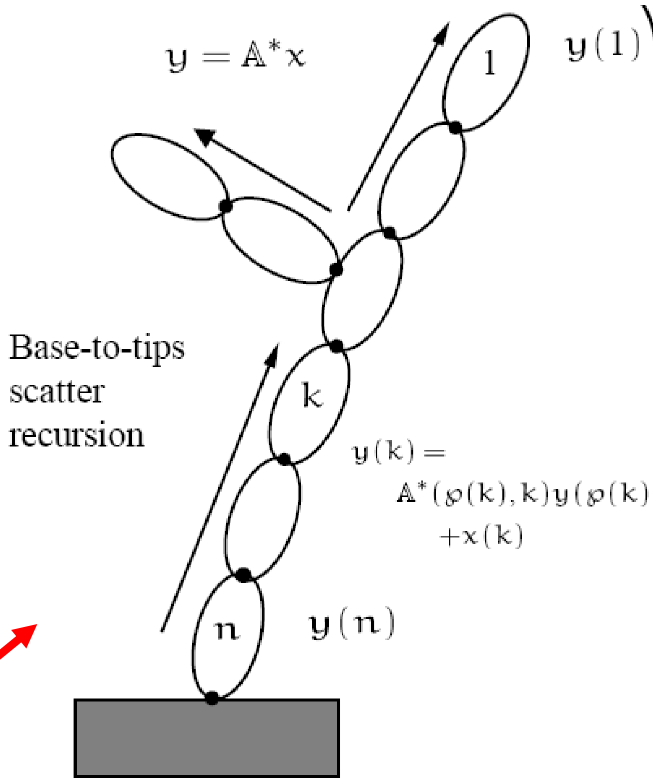
- *Applies to any x*
- *Does not require explicit computation of* $\mathbb{A}$ *at all*
- *Only depends on elements of* $\mathcal{E}_{\mathbb{A}}$

for *all nodes* $\mathbf{k}$ *(base-to-tips scatter)*

$$y(k) = \mathbb{A}^*(\wp(k), k)\, y(\wp(k)) + x(k)$$

end loop

O(N) structure-based scatter algorithm



Algorithm flow

Gather Recursions for ψ same as for ϕ

Recall: Tips-to-base structure-based O(N) <u>gather</u> recursion for ϕ

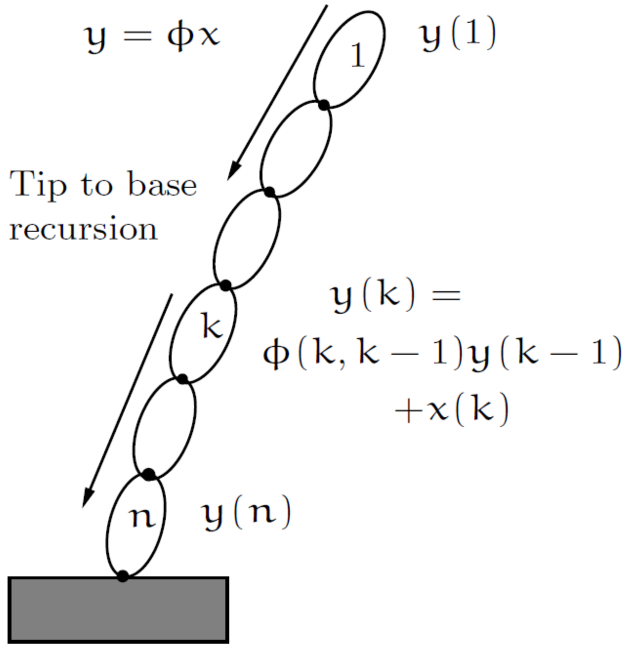
operator/vector product

$$y = \phi x$$

- *Applies to any x*
- *Does not require explicit computation of ϕ at all*
- *Only depends on elements of $\mathcal{E}_\phi$*

$$\begin{cases} y(0) = 0 \\ \textbf{for } k \quad 1 \cdots n \\ \quad y(k) = \phi(k, k-1) y(k-1) + x(k) \\ \textbf{end loop} \end{cases}$$

O(N) structure-based tip-to-base <u>gather</u> recursion



Algorithm flow

Example – torque for end-effector force

Generalized tips-to-base structure-based gather recursion

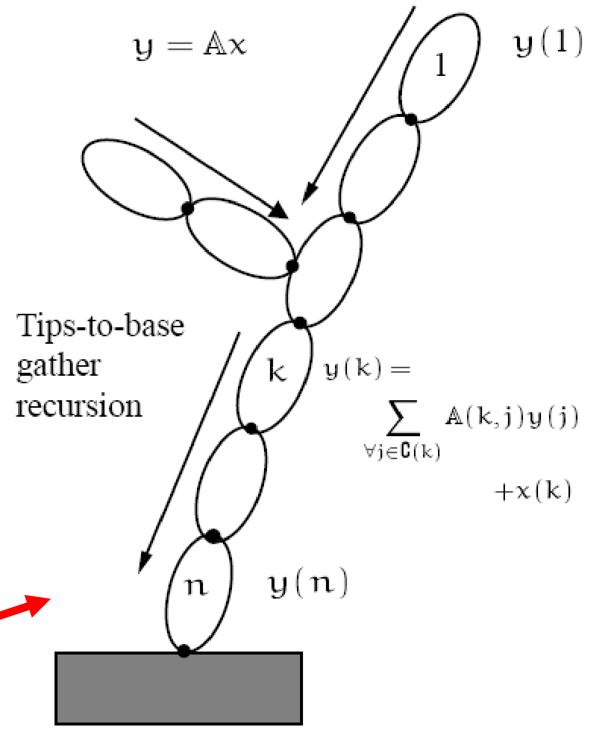
operator/vector product

$$\mathbf{y} = \mathbb{A}\mathbf{x}$$

$$\mathbb{A} = \phi,\ \psi$$

- *Applies to any x*
- *Does not require explicit computation of* $\mathbb{A}$ *at all*
- *Only depends on elements of* $\mathcal{E}_A$

$$\begin{cases} \textbf{for } \textit{all nodes } \mathrm{k} \quad \textit{(tips-to-base gather)} \\ \quad \mathrm{y}(\mathrm{k}) = \sum_{\forall i \in \mathbf{C}(k)} \mathbb{A}(\mathrm{k}, \mathrm{i})\mathrm{y}(\mathrm{i}) + \mathrm{x}(\mathrm{k}) \\ \textbf{end loop} \end{cases}$$



Algorithm flow

O(N) structure-based gather algorithm

ATBI Riccati Equation

Recall: Forward Lyapunov Equation for CRBs

CRB recursion

$$\mathcal{R}(k) = \phi(k, k-1)\mathcal{R}(k-1)\phi^*(k, k-1) + M(k)$$

Define CRB spatial operator

$$\mathcal{R} \triangleq \text{diag}\left\{\mathcal{R}(k)\right\}_{k=1}^{n} \in \mathcal{R}^{6n \times 6n}$$

Can re-express as CRB "forward Lyapunov equation" using spatial operators

$$\mathbf{M} = \mathcal{R} - \mathcal{E}_\phi \mathcal{R} \mathcal{E}_\phi^*$$

Riccati equation for ATBI

Similar to Lyapunov equation

$$\mathbf{M} = \mathcal{R} - \mathcal{E}_{\phi}\mathcal{R}\mathcal{E}_{\phi}^{*}$$

The ATBI recursion

$$\mathcal{P}(k+1) \overset{6.32}{=} \psi(k+1,k)\mathcal{P}(k)\psi^{*}(k+1,k) + \mathbf{M}(k+1)$$

can be re-expressed at the operator level as

$$\mathbf{M} = \mathcal{P} - \mathcal{E}_{\psi}\mathcal{P}\mathcal{E}_{\psi}^{*} = \mathcal{P} - \mathcal{E}_{\psi}\mathcal{P}\mathcal{E}_{\phi}^{*}$$

ATBI operator identity

Claim:

$$\mathcal{E}_\psi \mathcal{P} \mathcal{E}_\psi^* = \mathcal{E}_\psi \mathcal{P} \mathcal{E}_\phi^*$$

Derivation:

Have $\mathcal{P}(k)\overline{\tau}^*(k) = \overline{\tau}(k)\mathcal{P}(k) = \overline{\tau}(k)\mathcal{P}(k)\overline{\tau}^*(k)$

$$\mathcal{E}_\psi \mathcal{P} \mathcal{E}_\psi^* \overset{7.3}{=} \mathcal{E}_\psi \mathcal{P} \overline{\tau}^* \mathcal{E}_\phi^* \overset{7.3}{=} \mathcal{E}_\psi \overline{\tau} \mathcal{P} \mathcal{E}_\phi^* \overset{7.3}{=} \mathcal{E}_\phi \overline{\tau}\, \overline{\tau} \mathcal{P} \mathcal{E}_\phi^* = \mathcal{E}_\psi \mathcal{P} \mathcal{E}_\phi^*$$

Recall: Operator decomposition of $\phi \mathbf{M} \phi^*$

Claim:

$$\phi \mathbf{M} \phi^* = \mathcal{R} + \tilde{\phi}\mathcal{R} + \mathcal{R}\tilde{\phi}^*$$

Derivation:

$$\mathbf{M} = \mathcal{R} - \mathcal{E}_\phi \mathcal{R} \mathcal{E}_\phi^*$$

and thus pre & post multiplying

use identity

$$\tilde{\phi} \triangleq \phi - \mathbf{I} = \mathcal{E}_\phi \phi$$

$$\phi \mathbf{M} \phi^* \overset{4.9}{=} \phi \mathcal{R} \phi^* - \phi \mathcal{E}_\phi \mathcal{R} \mathcal{E}_\phi^* \phi^* \overset{3.41}{=} \phi \mathcal{R} \phi^* - \tilde{\phi} \mathcal{R} \tilde{\phi}^*$$

$$\overset{3.40}{=} (\tilde{\phi} + \mathbf{I}) \mathcal{R} (\tilde{\phi} + \mathbf{I}) - \tilde{\phi} \mathcal{R} \tilde{\phi}^* \overset{3.40}{=} \mathcal{R} + \tilde{\phi} \mathcal{R} + \mathcal{R} \tilde{\phi}^*$$

Operator decomposition of $\psi \mathbf{M} \psi^*$

Previously $\phi \mathbf{M} \phi^* = \mathcal{R} + \tilde{\phi}\mathcal{R} + \mathcal{R}\tilde{\phi}^*$

Claim:

disjoint decomposition

$$\psi \mathbf{M} \psi^* = \mathcal{P} + \tilde{\psi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*$$

block-diagonal

strictly lower triangular

strictly upper triangular

Derivation:

use $\mathbf{M} = \mathcal{P} - \mathcal{E}_\psi \mathcal{P} \mathcal{E}_\psi^*$

$$\psi \mathbf{M} \psi^* = \psi \mathcal{P} \psi^* - \psi \mathcal{E}_\psi \mathcal{P} \mathcal{E}_\psi^* \psi^* \overset{7.8}{=} (\tilde{\psi} + \mathbf{I})\mathcal{P}(\tilde{\psi} + \mathbf{I})^* - \tilde{\psi}\mathcal{P}\tilde{\psi}^*$$

$$= \mathcal{P} + \tilde{\psi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*$$

$$\tilde{\psi} \triangleq \psi \mathcal{E}_\psi = \mathcal{E}_\psi \psi = \psi - \mathbf{I}$$

use identity

Operator Identities

Identities relating ϕ, ψ

Now we will establish a sequence of identities that illustrate a close relationship between the ϕ, ψ spatial operators.

Identity: $\psi^{-1} - \phi^{-1} = \mathcal{K}H$

Claim:

$$\psi^{-1} - \phi^{-1} = \mathcal{K}H$$

Derivation:

$$\tau = \mathcal{G}H$$

$$\psi^{-1} = \mathbf{I} - \mathcal{E}_\psi \overset{7.3}{=} \mathbf{I} - \mathcal{E}_\phi \overline{\tau} \overset{7.3}{=} (\mathbf{I} - \mathcal{E}_\phi) + \mathcal{E}_\phi \tau$$

$$\overset{3.36,7.3}{=} \phi^{-1} + \mathcal{E}_\phi \mathcal{G}H \overset{7.3}{=} \phi^{-1} + \mathcal{K}H$$

More identities

Claim:

$$\psi^{-1}\phi = \mathbf{I} + \mathcal{K}H\phi$$
$$\phi\psi^{-1} = \mathbf{I} + \phi\mathcal{K}H$$
$$\phi^{-1}\psi = \mathbf{I} - \mathcal{K}H\psi$$
$$\psi\phi^{-1} = \mathbf{I} - \psi\mathcal{K}H$$

Derivation:

These follow by pre & post multiplying the following identity by $\phi,\ \psi$

$$\psi^{-1} - \phi^{-1} = \mathcal{K}H$$

… and some more identities

Claim:

$$[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi = \mathrm{H}\psi$$
$$\phi\mathcal{K}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}] = \psi\mathcal{K}$$
$$[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\mathrm{H}\psi = \mathrm{H}\phi$$
$$\psi\mathcal{K}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}] = \phi\mathcal{K}$$

Derivation:

use identity

$$\psi\phi^{-1} = \mathbf{I} - \psi\mathcal{K}\mathrm{H}$$

For the first identity,

$$[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi = \mathrm{H}[\mathbf{I} - \psi\mathcal{K}\mathrm{H}]\phi \overset{7.11}{=} \mathrm{H}(\psi\phi^{-1})\phi = \mathrm{H}\psi$$

Identities recap

$$\psi^{-1} - \phi^{-1} = \mathcal{K}\mathrm{H}$$

$$\begin{aligned}
\psi^{-1}\phi &= \mathbf{I} + \mathcal{K}\mathrm{H}\phi \\
\phi\psi^{-1} &= \mathbf{I} + \phi\mathcal{K}\mathrm{H} \\
\phi^{-1}\psi &= \mathbf{I} - \mathcal{K}\mathrm{H}\psi \\
\psi\phi^{-1} &= \mathbf{I} - \psi\mathcal{K}\mathrm{H}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]\mathrm{H}\phi &= \mathrm{H}\psi \\
\phi\mathcal{K}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}] &= \psi\mathcal{K} \\
[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\mathrm{H}\psi &= \mathrm{H}\phi \\
\psi\mathcal{K}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}] &= \phi\mathcal{K}
\end{aligned}$$

These identities are very useful in transforming and simplifying operator expressions. We will see their use in a number of instances ahead.

Identity $\mathrm{H}\psi\mathbf{M}\psi^{*}\mathrm{H}^{*} = \mathfrak{D}$

Expression for $\mathcal{H}\psi\mathbf{M}\psi^*\mathcal{H}^*$

Claim:

$$\mathcal{H}\psi\mathbf{M}\psi^*\mathcal{H}^* = \mathcal{D}$$

Derivation:

$$\begin{aligned}
\mathcal{H}\psi\mathbf{M}\psi^*\mathcal{H}^* &\overset{7.7}{=} \mathcal{H}(\mathcal{P} + \tilde{\psi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*)\mathcal{H}^* \overset{7.3}{=} \mathcal{D} + \mathcal{H}\tilde{\psi}\mathcal{P}\mathcal{H}^* + \mathcal{H}\mathcal{P}\tilde{\psi}^*\mathcal{H}^* \\
&\overset{7.8}{=} \mathcal{D} + \mathcal{H}\psi\mathcal{E}_\psi\mathcal{P}\mathcal{H}^* + \mathcal{H}\mathcal{P}\mathcal{E}_\psi^*\psi^*\mathcal{H}^* \\
&\overset{7.3}{=} \mathcal{D} + \mathcal{H}\psi\mathcal{E}_\phi\overline{\tau}\mathcal{P}\mathcal{H}^* + \mathcal{H}\mathcal{P}\overline{\tau}^*\mathcal{E}_\phi^*\psi^*\mathcal{H}^* \\
&\overset{7.3}{=} \mathcal{D} + \mathcal{H}\psi\mathcal{E}_\phi\mathcal{P}^+\mathcal{H}^* + \mathcal{H}\mathcal{P}^+\mathcal{E}_\phi^*\psi^*\mathcal{H}^* \\
&\overset{6.28}{=} \mathcal{D}
\end{aligned}$$

Comparison of operator expressions

Earlier mass matrix expression

$$\mathcal{M}(\theta) \triangleq H\phi\mathbf{M}\phi^*H^*$$

dense

versus similar expression

block-diagonal

$$H\psi\mathbf{M}\psi^*H^* = \mathcal{D}$$

Complex product of spatial operators collapses into just D!

The only difference is the use of ψ instead of ϕ!

Mass Matrix Innovations Factorization

Recall: The Newton-Euler Factorization of the mass matrix $\mathcal{M}(\theta)$

$$\mathcal{M}(\theta) \triangleq H\phi\mathbf{M}\phi^*H^*$$

- Square, symmetric and positive definite
- Size is the number of degrees of freedom
- The mass matrix is configuration dependent
- Dense matrix for serial chain systems
  - key reason for its perceived "complexity"
- Maps generalized velocities to system kinetic energy
- **Not** all of the operators in the Newton-Euler factorization of the mass matrix are **square**
  - Will encounter other factorizations with square factors
- Elements of ϕ^*H^* are Kane's partial velocities

Innovations Factorization of the mass matrix $\mathcal{M}(\theta)$

Claim:

Innovations Factorization

$$\mathcal{M} = [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\,\mathcal{D}\,[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^*$$

Derivation:

use identity

$$[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\mathrm{H}\psi = \mathrm{H}\phi$$

$$\begin{aligned}
\mathcal{M} &\overset{5.25}{=} \mathrm{H}\phi\mathbf{M}\phi^*\mathrm{H}^* = \mathrm{H}(\phi\psi^{-1})\psi\mathbf{M}\psi^*(\phi\psi^{-1})^*\mathrm{H}^* \\
&\overset{7.11}{=} \mathrm{H}[\mathbf{I} + \phi\mathcal{K}\mathrm{H}]\psi\mathbf{M}\psi^*[\mathbf{I} + \phi\mathcal{K}\mathrm{H}]^*\mathrm{H}^* \\
&= [\mathbf{I} + \mathrm{H}\phi\mathcal{K}](\mathrm{H}\psi\mathbf{M}\psi^*\mathrm{H}^*)[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^* \overset{7.13}{=} [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\mathcal{D}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^*
\end{aligned}$$

use identity

$$\mathrm{H}\psi\mathbf{M}\psi^*\mathrm{H}^* = \mathcal{D}$$

Properties of the Innovations Factorization of the mass matrix

$$\mathcal{M} = [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\,\mathcal{D}\,[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^*$$

Lower triangular with identity along block-diagonal

Block-diagonal

Upper triangular with identity along block-diagonal

- In the Newton-Euler factorization of the mass matrix, not all the factors were square
- However all the factors are square in the Innovations Factorization
- Moreover, the factors have block-triangular and block-diagonal structure
- And as we will see, they are easy to invert!

Implications for forward dynamics

$$\mathcal{M} = [\mathbf{I} + H\phi\mathcal{K}]\,\mathcal{D}\,[\mathbf{I} + H\phi\mathcal{K}]^*$$

Lower triangular with identity along block-diagonal

Block-diagonal

Upper triangular with identity along block-diagonal

- Forward dynamics involves computing $\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$ With the NE factorization our options were limited to $O(\mathcal{N}^3)$ complexity
- The new factors however can be computed at $O(\mathcal{N}^2)$ cost
- More these factors can be used to compute $\ddot{\theta} = \mathcal{M}^{-1}(\mathcal{T} - \mathcal{C})$ at $O(\mathcal{N}^2)$ cost as well.
- This is progress - exploitation of the underlying structure has reduced computational costs – again!

Recall: Trace of the mass matrix

$$\mathcal{M} = H\mathcal{R}H^* + H\tilde{\phi}\mathcal{R}H^* + H\mathcal{R}\tilde{\phi}^*H^*$$

zero trace

General expression

$$\text{Trace}\{\mathcal{M}(\theta)\} = \sum_{i=1}^{n} \text{Trace}\{H(k)\mathcal{R}(k)H^*(k)\}$$

For 1 dof hinges

$$\text{Trace}\{H(k)\mathcal{R}(k)H^*(k)\} = H(k)\mathcal{R}(k)H^*(k)$$

Determinant of the mass matrix

$$\mathcal{M} = \left[\mathbf{I} + \mathrm{H}\phi\mathcal{K}\right]\mathcal{D}\left[\mathbf{I} + \mathrm{H}\phi\mathcal{K}\right]^*$$

Lower triangular with identity along block-diagonal

Block-diagonal

Upper triangular with identity along block-diagonal

determinant 1

The determinant of a matrix is the product of the determinant of its square factors

General expression

$$\det\{\mathcal{M}\} = \prod_{k=1}^{n} \det\{\mathcal{D}(k)\}$$

scalar for 1 dof hinge

Application of mass matrix determinant: Fixman Potential

The Fixman potential is needed in molecular dynamics simulations for correcting statistical biases

$$\mathcal{U}_f \triangleq \log\{\det\{\mathcal{M}\}\}$$

$$\prod_{k=1}^{n} \det\{\mathcal{D}(k)\}$$

Computing and using it has been an intractable problem for decades

$$\frac{\partial \log\{\det\{\mathcal{M}\}\}}{\partial \theta_i} = 2\,\text{Trace}\left\{\mathcal{P}(i)\Upsilon(i)\widetilde{H}^*(i)\right\}$$

Torque from the Fixman potential

Available from standard ATBI computations

Explicit simple expression via SOA for longstanding ***intractable*** *problem.*

More operator decompositions

Decomposition of $\phi \mathbf{M} \psi^*$

Previously $\psi \mathbf{M} \psi^* = \mathcal{P} + \tilde{\psi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*$

Claim:

$$\phi \mathbf{M} \psi^* = \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*$$

block-diagonal ($\mathcal{P}$), *strictly lower triangular* ($\tilde{\phi}\mathcal{P}$), *strictly upper triangular* ($\mathcal{P}\tilde{\psi}^*$)

Derivation:

Have $\mathbf{M} = \mathcal{P} - \mathcal{E}_\phi \mathcal{P} \mathcal{E}_\psi^*$

Pre and post multiply by ϕ, ψ to get

$$\phi \mathbf{M} \psi^* \overset{3.41,7.8}{=} \phi \mathcal{P} \psi^* - \tilde{\phi}\mathcal{P}\tilde{\psi}^* = (\tilde{\phi} + \mathbf{I})\mathcal{P}(\tilde{\psi}^* + \mathbf{I}) - \tilde{\phi}\mathcal{P}\tilde{\psi}^*$$
$$= \tilde{\phi}\mathcal{P}\tilde{\psi}^* + \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\psi}^* - \tilde{\phi}\mathcal{P}\tilde{\psi}^* = \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\psi}^*$$

Another decomposition of $\phi \mathbf{M} \phi^*$

Previously $\phi \mathbf{M} \phi^* = \mathcal{R} + \tilde{\phi}\mathcal{R} + \mathcal{R}\tilde{\phi}^*$ *using CRBs*

Claim:

using ATBIs

$$\phi \mathbf{M} \phi^* = \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\phi}^* + \phi\mathcal{K}\mathcal{D}\mathcal{K}^*\phi^*$$

block-diagonal *strictly lower triangular* *strictly upper triangular* *dense*

Derivation:

$$\phi \mathbf{M} \phi^* = (\phi\psi^{-1})\psi\mathbf{M}\phi^* \overset{7.15}{=} (\phi\psi^{-1})[\psi\mathcal{P} + \mathcal{P}\tilde{\phi}^*]$$

$$\overset{7.11}{=} \phi\mathcal{P} + [\mathbf{I} + \phi\mathcal{K}\mathrm{H}]\mathcal{P}\tilde{\phi}^* = \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\phi}^* + \phi\mathcal{K}\mathrm{H}\mathcal{P}\mathcal{E}_{\phi}^*\phi^*$$

$$\overset{6.13}{=} \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\phi}^* + \phi\mathcal{K}\mathcal{D}\mathcal{G}^*\mathcal{E}_{\phi}^*\phi^* \overset{6.36}{=} \mathcal{P} + \tilde{\phi}\mathcal{P} + \mathcal{P}\tilde{\phi}^* + \phi\mathcal{K}\mathcal{D}\mathcal{K}^*\phi^*$$

This is an alternative decomposition using ATBIs.

CRB vs ATBI comparison

Claim:

$$\mathcal{R} \geqslant \mathcal{P}.$$

SHOW!

Hint: *use results on previous slide*

Jet Propulsion Laboratory
California Institute of Technology

Inversion of the Mass Matrix

Inverting the mass matrix $\mathcal{M}(\theta)$

$$\mathcal{M} = [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\,\mathcal{D}\,[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^*$$

Lower triangular with identity along block-diagonal

Block-diagonal

Upper triangular with identity along block-diagonal

- All the factors are square in the Innovations Factorization
- So want to look into inverting the mass matrix by inverting its factors
- D is block-diagonal, and easy to invert
- We will thus focus on inverting $[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]$

Inverse of $[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]$

Claim:

$$[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^{-1} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]$$

Lower triangular with identity along block-diagonal

Derivation:

Have general matrix identity

$$(\mathbf{I} + \mathrm{AB})^{-1} = \mathbf{I} - \mathrm{A}(\mathbf{I} + \mathrm{BA})^{-1}\mathrm{B}$$

$$[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^{-1} = \mathbf{I} - \mathrm{H}[\mathbf{I} + \phi\mathcal{K}\mathrm{H}]^{-1}\phi\mathcal{K} \overset{7.11}{=} \mathbf{I} - \mathrm{H}(\phi\psi^{-1})^{-1}\phi\mathcal{K}$$

$$= \mathbf{I} - \mathrm{H}\psi\mathcal{K}$$

using $\phi\psi^{-1} = \mathbf{I} + \phi\mathcal{K}\mathrm{H}$

Mass matrix inverse

Claim:

$$\mathcal{M}^{-1} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^*\mathcal{D}^{-1}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]$$

Derivation:

$$\mathcal{M}^{-1} \overset{7.14}{=} \{[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]\mathcal{D}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^*\}^{-1} = [\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^{-*}\mathcal{D}^{-1}[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^{-1}$$

$$\overset{7.4}{=} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^*\mathcal{D}^{-1}[\mathbf{I} - \mathrm{H}\psi\mathcal{K}]$$

using $[\mathbf{I} + \mathrm{H}\phi\mathcal{K}]^{-1} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]$

Comments on mass matrix inverse

$$\mathcal{M}^{-1} = [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - \mathrm{H}\psi\mathcal{K}]$$

Upper triangular with identity along block-diagonal

Block-diagonal

Lower triangular with identity along block-diagonal

- Analytical, closed-form expression for the mass matrix using spatial operators
- The factors are square and invertible – and have diagonal and triangular structure
- The expression is valid for any size system and branching structure

Progression with the Mass Matrix

$$\mathcal{M} = H\phi M\phi^* H^*$$

Analytical Newton-Euler factorization of the mass matrix

$$\mathcal{M} = [I + H\phi\mathcal{K}]\mathcal{D}[I + H\phi\mathcal{K}]^*$$

Analytical Innovations factorization of the mass matrix

$$[I + H\phi\mathcal{K}]^{-1} = I - H\psi\mathcal{K}$$

$$\mathcal{M}^{-1} = [\mathbf{I} - H\psi\mathcal{K}]^* \mathcal{D}^{-1} [\mathbf{I} - H\psi\mathcal{K}]$$

Analytical operator expression for the mass matrix inverse

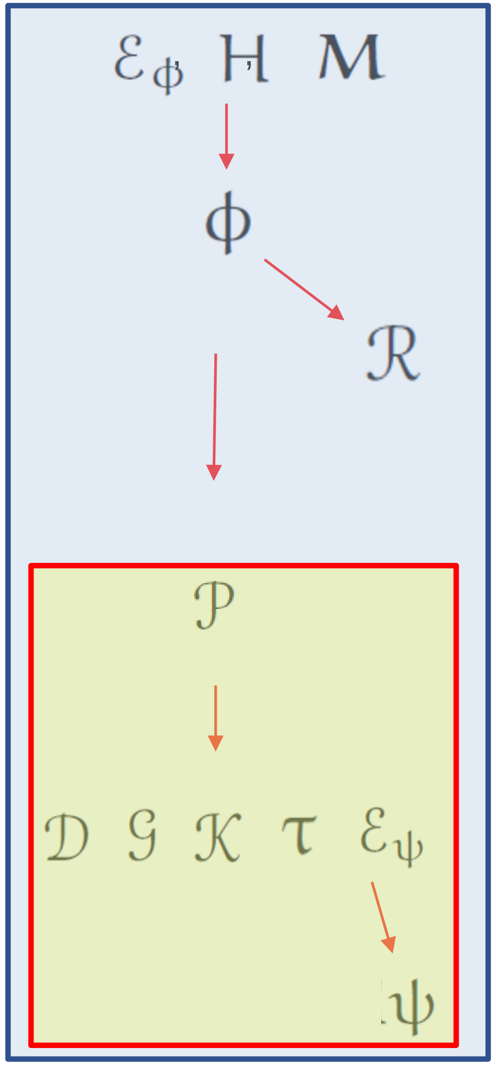
How far have we come?

Spatial operators

- Velocity expression
- Jacobian
- Mass matrix NE factorization
- Lyapunov equation for CRBs
- Mass matrix decomposition
- Riccati equation for ATBI
- Several operator identities
- Mass matrix Innovations factorization
- Mass matrix determinant
- Mass matrix inverse and factorization

Have started to build up a vocabulary of spatial operators that can be used to express and manipulate the structure of dynamics quantities.

*Now can see the rationale for the **algebra** part of SOA from the analytical transformations and simplifications possible using the operators.*



spatial operators family

Recursive Computational Algorithms

- O(N) Gather and scatter recursions pattern
- O(N) Body velocities scatter recursion
- O(N) CRBs gather recursion
- $O(\mathcal{N}^2)$ mass matrix computation
- O(N) NE scatter/gather inverse dynamics
- $O(\mathcal{N}^2)$ inverse dynamics based mass matrix
- O(N) CRBs based inverse dynamics
- O(N) ATBI gather recursion
- $O(\mathcal{N}^2)$ forward dynamics

Can derive such low-cost scatter/gather algorithms usually by examination of the spatial operator expressions.

Summary

- Introduced ATBI spatial operators
- Developed several operator identities
- Developed Innovations Factorization of the mass matrix
  - Has square and invertible factors
  - Can reduce forward dynamics costs
- Developed expression for inverse of factors
- Developed operator expression for the mass matrix inverse

SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

SOA Generalization Track Topics

8. **Graph theory based structure** – BWA matrices, connection to multibody systems
9. **Tree topology systems** – generalization to tree topology rigid body systems, SKO/SPO operators, gather/scatter algorithms
10. **Closed-chain dynamics (cut-joint)** – holonomic and non-holonomic constraints, cut-joint method, operational space inertia, projected dynamics
11. **Closed-chain dynamics (constraint embedding)** – constraint embedding for graph transformation, minimal coordinate closed-chain dynamics
12. **Flexible body dynamics** – Extension to flexible bodies, modal representations, recursive flexible body dynamics

SOA Generalization Track Topics

1. **Graph theory based structure** – BWA matrices, SKO and SPO operators
2. **Tree topology systems** – generalization to tree topology rigid body systems, gather/scatter algorithms
3. **Operational space dynamics** – dynamics in task/constraint space, operational space inertia, decomposition and recursive computation
4. **Closed-chain dynamics (cut-joint)** – holonomic and non-holonomic constraints, cut-joint method, move and squeeze forces, projected dynamics
5. **Graph transformations –** Multibody topology transformation and decomposition, aggregation
6. **Closed-chain dynamics (constraint embedding)** – constraint embedding for graph transformation, minimal coordinate closed-chain dynamics
7. **Flexible body dynamics** – Extension to flexible bodies, modal representations, recursive flexible body dynamics
8. **Numerical methods and integrators** (Radu Serban) – ODE and DAE integration for multibody dynamics