Dynamics and Real-Time Simulation (DARTS) Laboratory

Spatial Operator Algebra (SOA)

1. Articulated Body Inertias

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June 19, 2024

https://dartslab.jpl.nasa.gov/

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SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

See https://dartslab.jpl.nasa.gov/References/index.php for publications and references on the SOA methodology.

Recap

Recap

- Developed Newton-Euler factorization of the mass matrix
- Introduced CRB inertias for the decomposition of the mass matrix and its $O(\mathcal{N}^2)$ computation
- Developed operator form of system equations of motion
- Developed O(N) Newton-Euler inverse dynamics algorithm
- Explored inverse dynamics based computation of mass matrix, and CRB based inverse dynamics and force decompositions

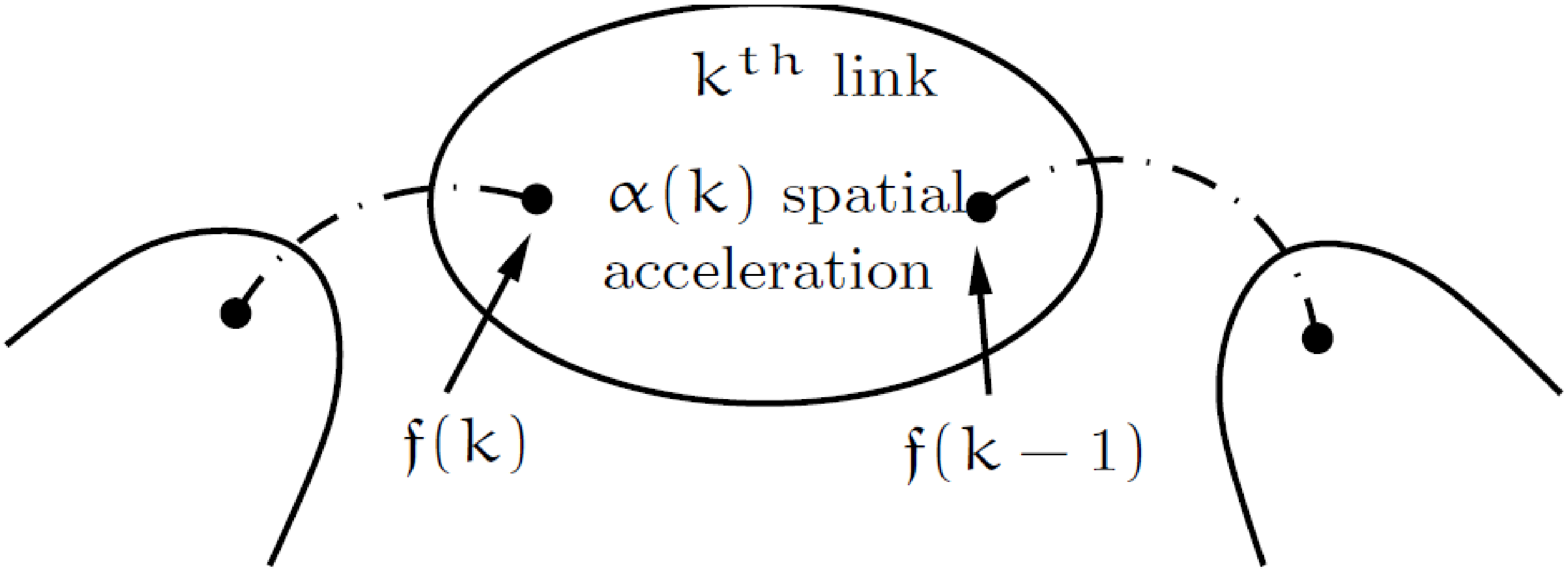
Background

- We now switch to the **forward dynamics** problem which involves the mass matrix inverse
- While we can directly go the spatial operator route, we will take a step back to work at the component level to build up some physical intuition
- This route will involve a new quantity referred to as the ***articulated body inertia***
  - This and related quantities are the focus of this session

Inter-body force decompositions

Inter-body spatial force decomposition models

Terminal and **Composite Rigid Body** models



Terminal-body model: $\mathfrak{f}(k) = \mathfrak{M}(k)\alpha(k) + \mathfrak{f}(k-1)$

Composite-body model: $\mathfrak{f}(k) = \mathcal{R}(k)\alpha(k) + \mathfrak{y}(k)$

inertia term

residual force term

Inter-body spatial force decompositions

- Ignore Coriolis terms for the moment
- Force decompositions consist of inertia + residual terms
- From the equations of motion we had

$$\mathfrak{f}(k) = \underbrace{M(k)}_{\text{depends on } \textbf{kth} \text{ body}}\alpha(k) + \phi(k, k-1)\underbrace{\mathfrak{f}(k-1)}_{\text{depends on } \textbf{all} \text{ bodies}}$$

*depends on **kth** body*

*depends on **all** bodies*

- Using CRBs we have the alternative expression

$$\mathfrak{f}(k) = \underbrace{\mathcal{R}(k)}_{\text{depends on } \textbf{outboard} \text{ bodies only}}\alpha(k) + \underbrace{\mathfrak{y}(k)}_{\text{depends on } \textbf{outboard} \text{ generalized accels}}$$

*depends on **outboard** bodies only*

*depends on **outboard** generalized accels*

- The more complex inertia term simplifies the residual force term in the force decompositions
- We will see more such decompositions later

Why force decompositions?

- Force decompositions provide an opportunity to view the dynamics in new ways
  - View the general multibody system as a deviation from a reference model
- Composite body inertias provide an alternative way to describe the accels/force relationship and additional insight
- The decompositions also provide a pathway to computational algorithms, eg. the CRBs based inverse dynamics algorithms
- The articulated body force decomposition we pursue here will provide the basis for the O(N) recursive forward dynamics algorithm

Single body equations of motion decomposition

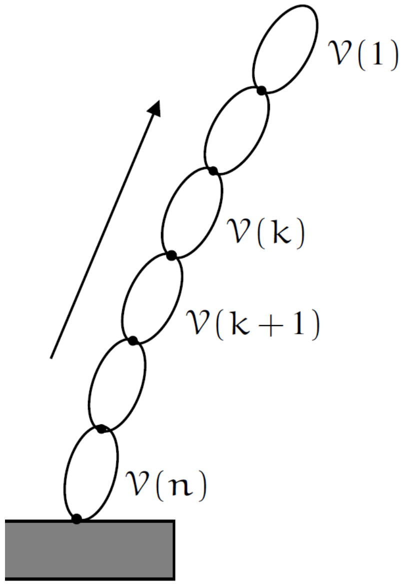
Terminal body model

- Residual term is zero when the kth body is a terminal body. We then have the simpler relationship

$$\mathfrak{f}(k) = \mathcal{M}(k)\alpha(k)$$

- In reality, there are outboard bodies, and the residual term accounts for the interaction with all the outboard bodies

$$\mathfrak{f}(k) = \mathcal{M}(k)\alpha(k) + \boxed{\phi(k, k-1)\mathfrak{f}(k-1)}$$



CRB decomposition

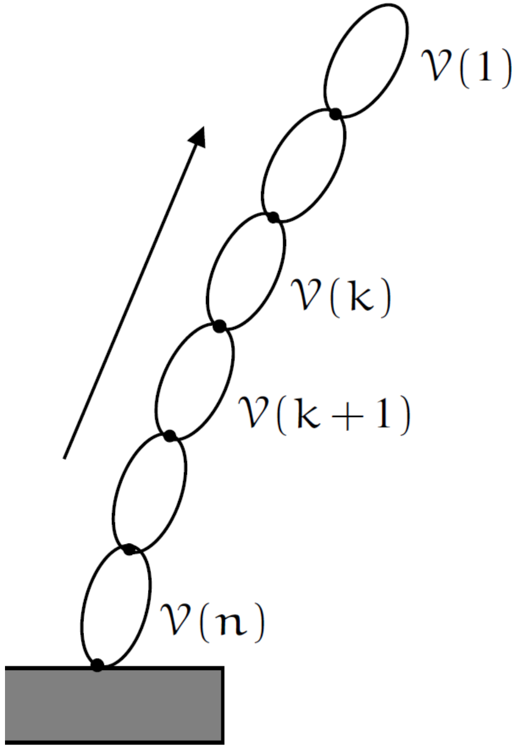
Composite body model

- An improvement over terminal body model in not ignoring outboard bodies
- Residual term is zero if outboard generalized accels are zero, i.e. if outboard bodies are rigid

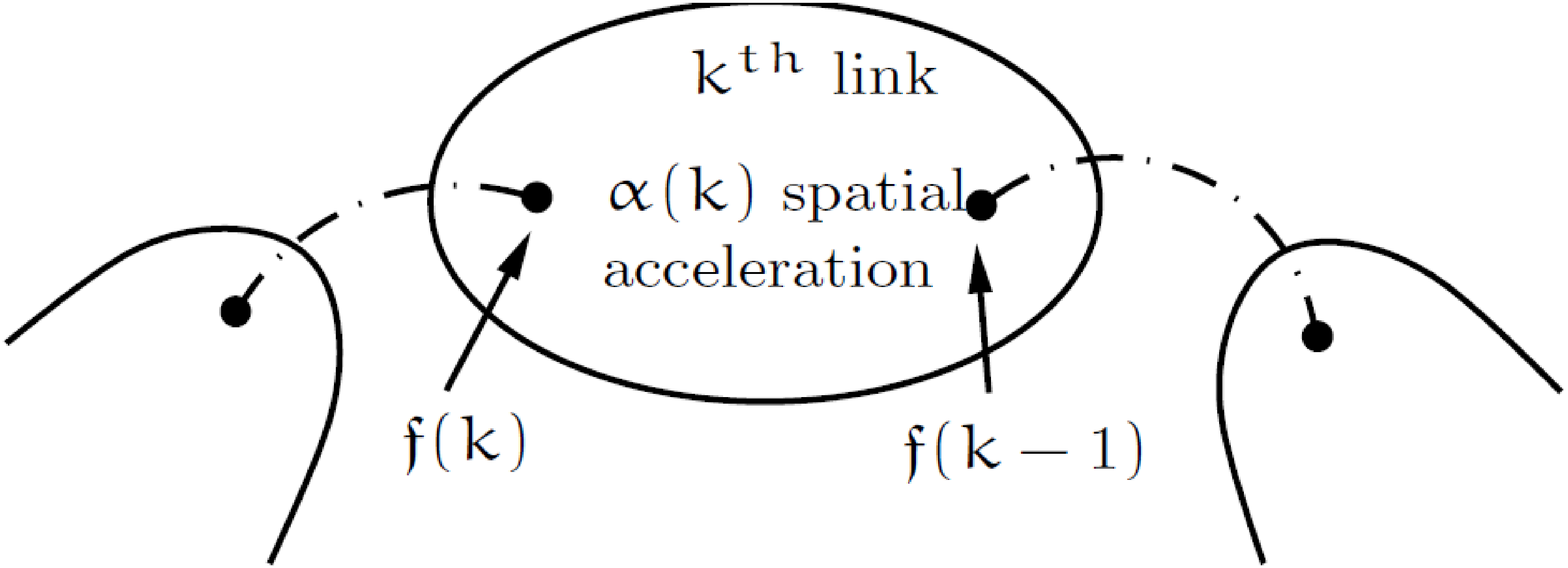
$$\mathfrak{f}(k) = \mathcal{R}(k)\alpha(k)$$

- The residual term accounts for the non-zero generalized accels of the outboard bodies
  - The gen accels of the inboard bodies does not matter

$$\mathfrak{f}(k) = \mathcal{R}(k)\alpha(k) + \boxed{\mathfrak{y}(k)}$$



Inter-body force decomposition models



Terminal-body model: $\mathfrak{f}(k) = \mathcal{M}(k)\alpha(k) + \mathfrak{f}(k-1)$

Composite-body model: $\mathfrak{f}(k) = \mathcal{R}(k)\alpha(k) + \mathfrak{y}(k)$

$\boxed{\text{Articulated-body model: } \mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)}$

Will now develop this model …

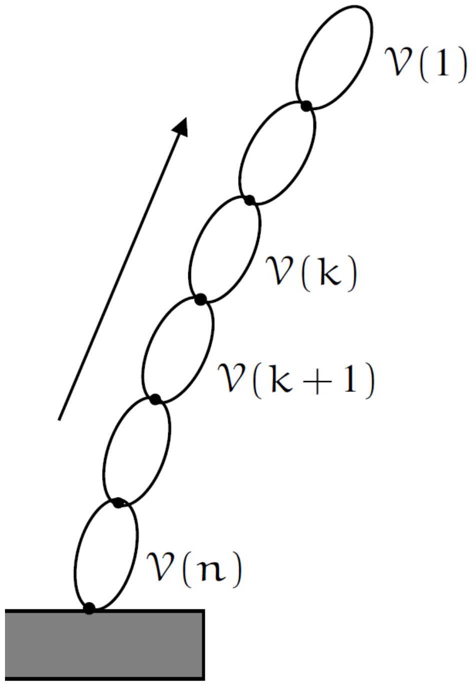
Floppy Articulated body model

Articulated Body ‘floppy’ model

Lets first focus on the “floppy” case, i.e. where the outboard hinges are free with zero generalized forces. Later will allow non-zero generalized forces.

Question

What is the effective inertia, i.e. the force/acceleration relationship at the (k+1)th body when the outboard bodies are floppy, i.e. the outboard body generalized forces are all 0?



Tip body's articulated body inertia

- Clearly know the answer for the tip body

$$\mathfrak{f}(1) = \mathcal{M}(1)\alpha(1)$$

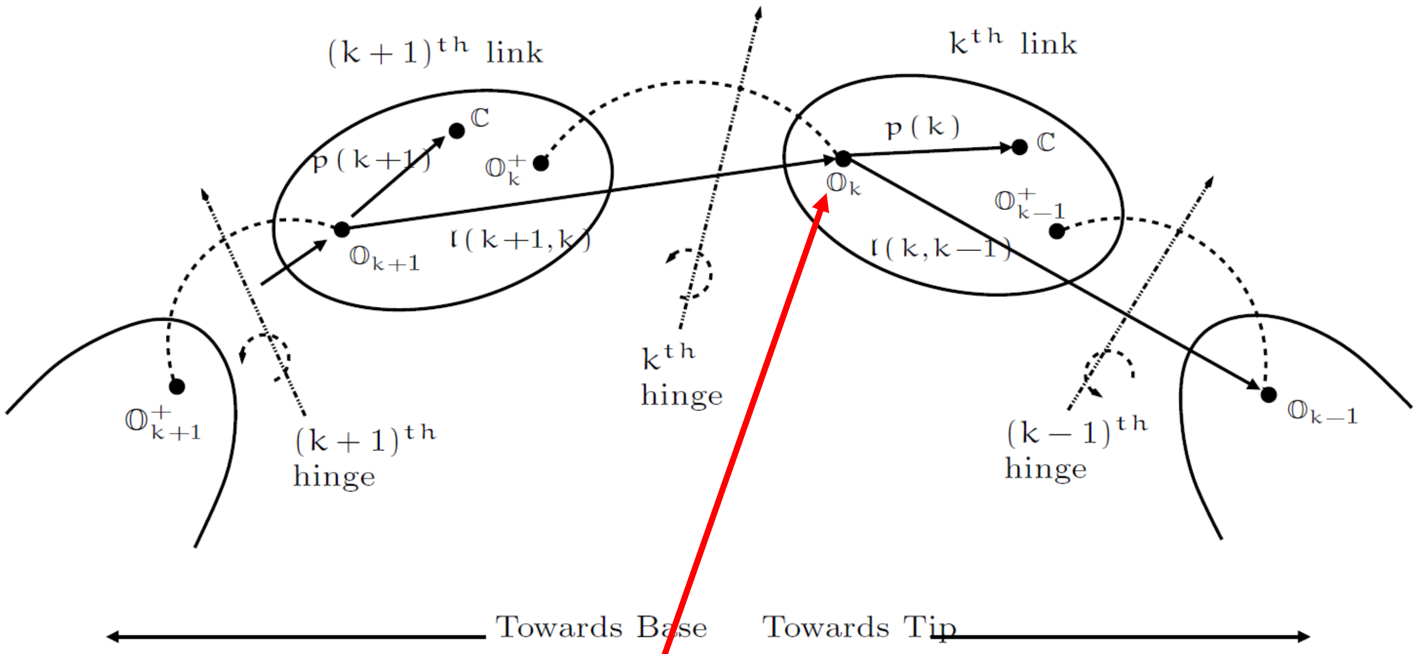
articulated body inertia (ATBI)

$$\boxed{\mathcal{P}(1) = \mathcal{M}(1)}$$

- Will use induction based argument to extend to other bodies
- So let us assume we have established the ATBI relationship for the kth body

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k)$$

Induction based derivation - start

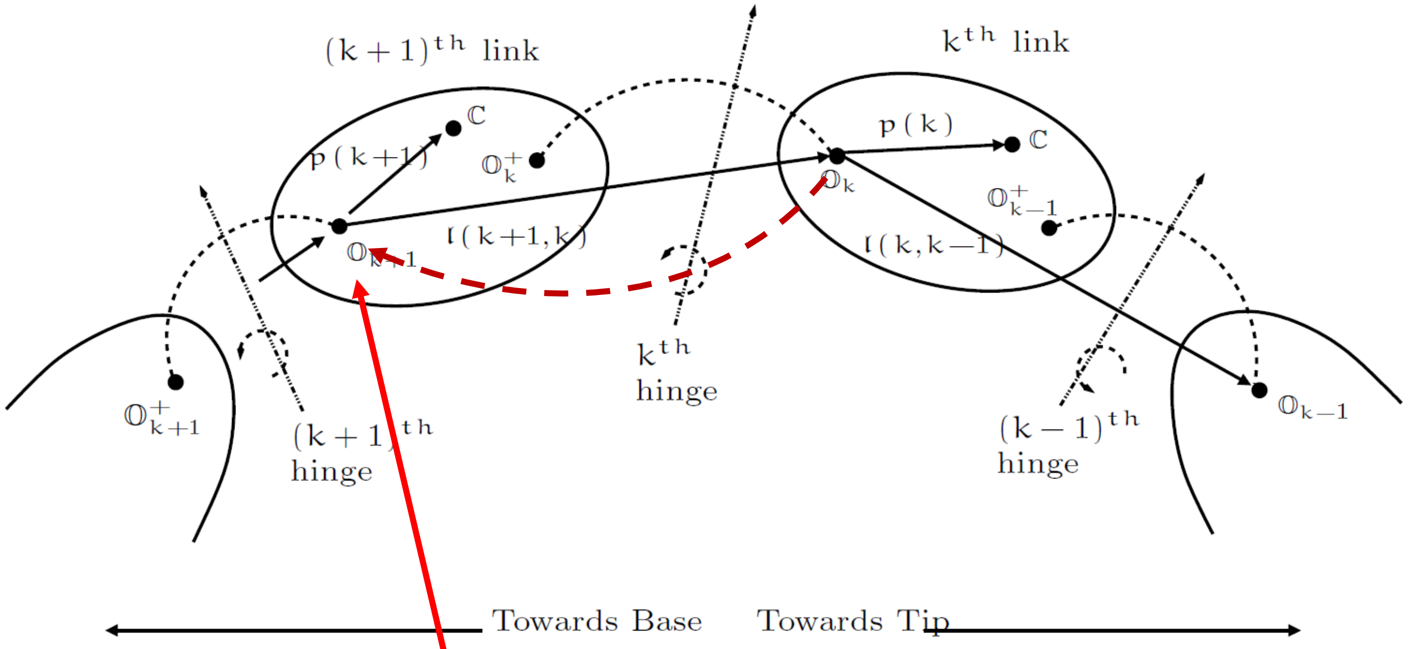


Assume we know the relationship for the kth body:

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k)$$

where $\mathcal{T}(j) = H(j)\mathfrak{f}(j) = \mathbf{0} \quad \forall\, j < k$

Induction based derivation - end



Want to establish the relationship for body (k+1)

$$f(k+1) = \mathcal{P}(k+1)\alpha(k+1)$$

$$\mathcal{T}(j) = H(j)f(j) = \mathbf{0} \quad \forall\, j < \boxed{k+1}$$

Conditions to be met

$$\boxed{\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k)} \qquad \boxed{\mathcal{T}(j) = H(j)\mathfrak{f}(j) = \mathbf{0} \quad \forall\, j < k+1}$$

Condition to be met for kth hinge

- Hinge k (connecting bodies k and (k+1)) is free, i.e. its generalized force is 0, i.e.

$$\mathbf{0} \overset{6.7}{=} \mathcal{T}(k) \overset{6.7}{=} H(k)\mathfrak{f}(k) = H(k)\mathcal{P}(k)\alpha(k)$$

Acceleration relationships

Know $\alpha(k+1)$, need to find $\mathfrak{f}(k+1)$ & $\mathcal{P}(k+1)$

Have $$\alpha^{+}(k) = \phi^{*}(k+1,k)\alpha(k+1)$$

and $$\alpha(k) = \alpha^{+}(k) + H^{*}(k)\ddot{\theta}(k)$$

$$\begin{aligned}\mathbf{0} \overset{6.7}{=} \mathcal{T}(k) &\overset{6.7}{=} H(k)\mathfrak{f}(k) = H(k)\mathcal{P}(k)\alpha(k)\\ &\overset{6.10}{=} H(k)\mathcal{P}(k)\alpha^{+}(k) + H(k)\mathcal{P}(k)H^{*}(k)\ddot{\theta}(k)\end{aligned}$$

Value of $\ddot{\theta}(k)$ induced by $\alpha(k+1)$

Solving for generalized accel at the hinge

$$\mathbf{0} \overset{6.7}{=} \mathcal{T}(k) \overset{6.7}{=} H(k)\mathfrak{f}(k) = H(k)\mathcal{P}(k)\alpha(k)$$

$$\overset{6.10}{=} H(k)\mathcal{P}(k)\alpha^{+}(k) + H(k)\mathcal{P}(k)H^{*}(k)\ddot{\theta}(k)$$

$$\Downarrow$$

$$\boxed{\ddot{\theta}(k) \overset{6.11}{=} -\mathcal{D}^{-1}(k)H(k)\mathcal{P}(k)\alpha^{+}(k) = -\mathcal{G}^{*}(k)\alpha^{+}(k)}$$

where

$$\boxed{\mathcal{D}(k) \overset{\triangle}{=} H(k)\mathcal{P}(k)H^{*}(k)} \qquad \boxed{\mathcal{G}(k) \overset{\triangle}{=} \mathcal{P}(k)H^{*}(k)\mathcal{D}^{-1}(k)}$$

Identity

From the definitions

$$\mathcal{D}(k) \triangleq H(k)\mathcal{P}(k)H^*(k)$$

$$\mathcal{G}(k) \triangleq \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k)$$

It follows that

$$\boxed{H(k)\mathcal{G}(k) = \mathbf{I}}$$

Value of $\alpha(k)$ induced by $\alpha(k+1)$

$$\boxed{\ddot{\theta}(k) \overset{6.11}{=} -\mathcal{D}^{-1}(k)H(k)\mathcal{P}(k)\alpha^{+}(k) = -\mathcal{G}^{*}(k)\alpha^{+}(k)}$$

With $\alpha(k) \overset{5.21,6.9}{=} \alpha^{+}(k) + H^{*}(k)\ddot{\theta}(k)$

have

$$\alpha(k) \overset{6.10,6.12}{=} [\mathbf{I} - H^{*}(k)\mathcal{G}^{*}(k)]\alpha^{+}(k) = \boxed{\overline{\tau}^{*}(k)\alpha^{+}(k)}$$

where

$$\boxed{\tau(k) \overset{\triangle}{=} \mathcal{G}(k)H(k)} \qquad \boxed{\overline{\tau}(k) \overset{\triangle}{=} \mathbf{I} - \tau(k) = \mathbf{I} - \mathcal{G}(k)H(k)}$$

$\tau(k)$ and $\overline{\tau}(k)$ are projections

Claim: $\tau(k)$ is a projection

Proof:

Have $\mathcal{H}(k)\mathcal{G}(k) = \mathbf{I}$

and $\tau(k) \triangleq \mathcal{G}(k)\mathcal{H}(k)$

$$\tau(k) \cdot \tau(k) \overset{6.16}{=} \mathcal{G}(k)\mathcal{H}(k)\mathcal{G}(k)\mathcal{H}(k) \overset{6.14}{=} \mathcal{G}(k)\mathcal{H}(k) = \tau(k)$$

Since $\overline{\tau}(k) \triangleq \mathbf{I} - \tau(k)$ it is a projection too.

$\tau(k)$ and $\overline{\tau}(k)$ projection properties

Identities:

$$\boxed{\tau(k)\mathcal{G}(k) = \mathcal{G}(k)}$$

$$\boxed{\overline{\tau}^*(k)H^*(k) = \mathbf{0}}$$

SHOW!

More $\tau(k)$ projection properties

Claim:

$$\boxed{\tau(k)\mathcal{P}(k) = \mathcal{P}(k)\tau^*(k) = \tau(k)\mathcal{P}(k)\tau^*(k)}$$

$$\boxed{\mathcal{P}(k)\overline{\tau}^*(k) = \overline{\tau}(k)\mathcal{P}(k) = \overline{\tau}(k)\mathcal{P}(k)\overline{\tau}^*(k)}$$

SHOW!

Proof: (of first identity)

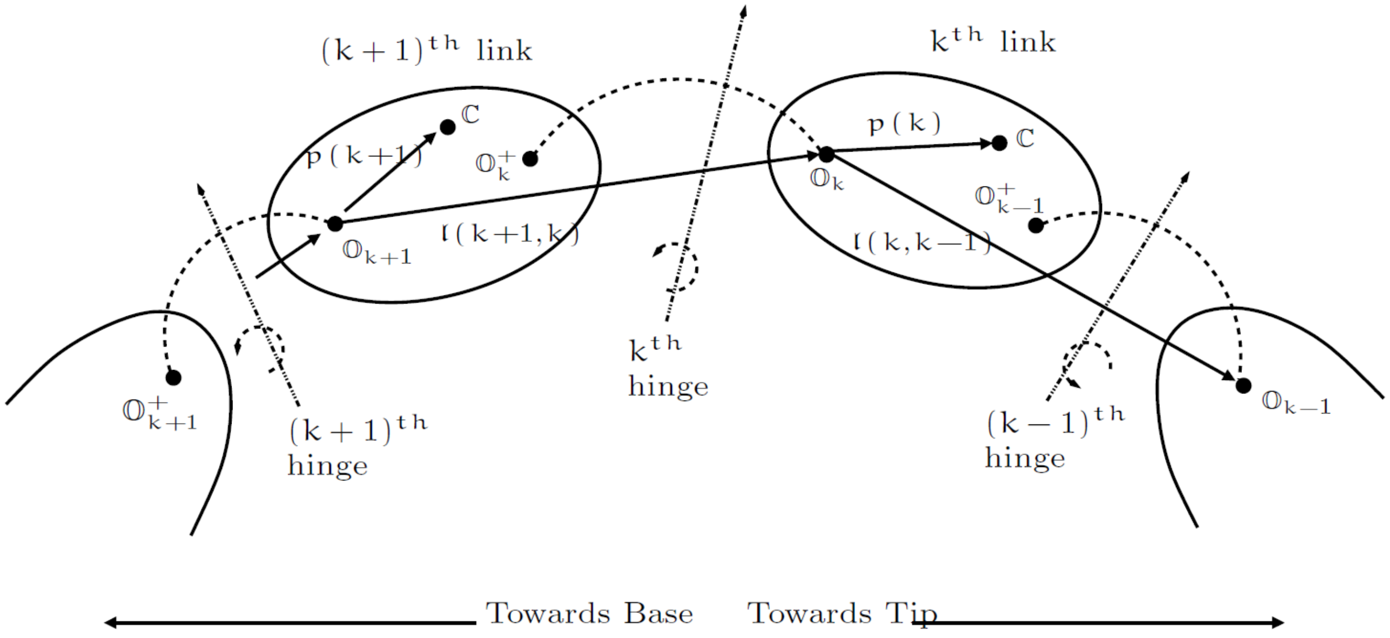
Using $\tau(k) \triangleq \mathcal{G}(k)H(k)$ & $\mathcal{G}(k) \triangleq \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k)$

we have

$$\tau(k)\mathcal{P}(k) \overset{6.13,6.16}{=} \mathcal{P}(k)H^*(k)\mathcal{D}^{-1}(k)H(k)\mathcal{P}(k) \overset{6.13,6.16}{=} \mathcal{P}(k)\tau^*(k)$$

Also $\tau(k)\mathcal{P}(k) = [\tau(k)]^2\mathcal{P}(k) \overset{6.27}{=} \tau(k)\mathcal{P}(k)\tau^*(k)$

Projection of $\alpha^{+}(k)$



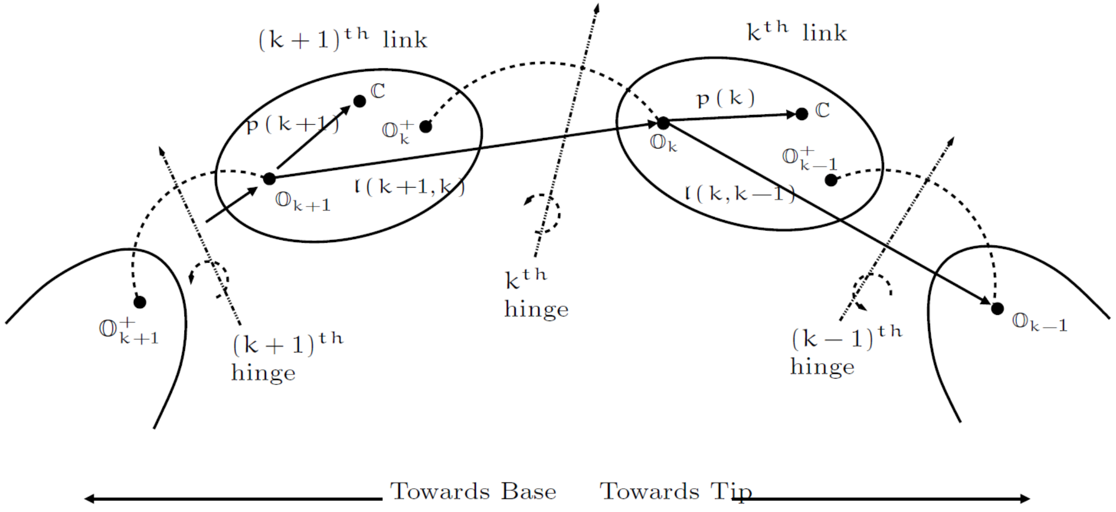
Recall: $\alpha(k) = \alpha^{+}(k) + H^{*}(k)\ddot{\theta}(k)$

$$\boxed{\overline{\tau}^{*}(k)H^{*}(k) = \mathbf{0}} \Rightarrow \boxed{\alpha(k) = \overline{\tau}^{*}(k)\alpha^{+}(k)}$$

$\overline{\tau}^{*}(k)$ nullifies hinge accels for a floppy hinge

$\overline{\tau}^{*}(k)$ removes hinge contribution when transmitting accel across the floppy hinge

Crossing the hinge with $\mathcal{P}(k)$



Have: $\alpha(k) = \overline{\tau}^*(k)\alpha^+(k)$

$$f(k) \overset{6.7}{=} \mathcal{P}(k)\alpha(k) \overset{6.15}{=} \boxed{\mathcal{P}(k)\overline{\tau}^*(k)}\alpha^+(k) = \mathcal{P}^+(k)\alpha^+(k)$$

ATBI on the inboard side of the hinge

$$\boxed{\mathcal{P}^+(k) \triangleq \mathcal{P}(k)\overline{\tau}^*(k)}$$

Identity $\mathcal{P}^+(\mathrm{k})$

Have:

$$\boxed{\mathcal{P}(\mathrm{k})\overline{\tau}^*(\mathrm{k}) = \overline{\tau}(\mathrm{k})\mathcal{P}(\mathrm{k}) = \overline{\tau}(\mathrm{k})\mathcal{P}(\mathrm{k})\overline{\tau}^*(\mathrm{k})}$$

$$\boxed{\mathcal{P}^+(\mathrm{k}) \triangleq \mathcal{P}(\mathrm{k})\overline{\tau}^*(\mathrm{k})}$$

$$\Downarrow$$

$$\boxed{\mathcal{P}^+(\mathrm{k}) = \overline{\tau}(\mathrm{k})\mathcal{P}(\mathrm{k}) = \mathcal{P}(\mathrm{k})\overline{\tau}^*(\mathrm{k}) = \overline{\tau}(\mathrm{k})\mathcal{P}(\mathrm{k})\overline{\tau}^*(\mathrm{k})}$$

ATBI Expression $\mathcal{P}(k+1)$

Claim:

ATBI for (k+1) body

$$\boxed{\mathcal{P}(k+1) \overset{\triangle}{=} \phi(k+1,k)\mathcal{P}^{+}(k)\phi^{*}(k+1,k) + M(k+1)}$$

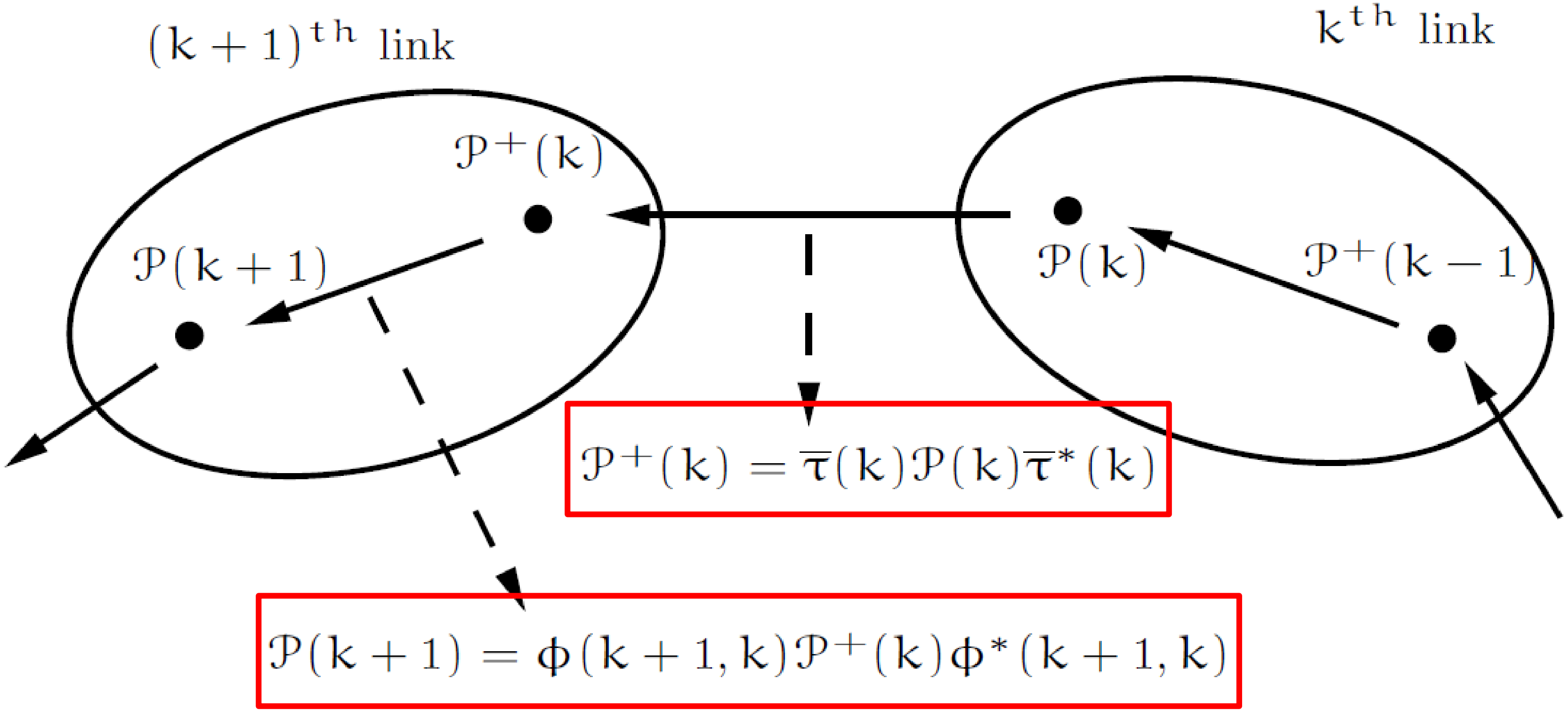
Proof:

Using $\alpha^{+}(k) = \phi^{*}(k+1,k)\alpha(k+1)$ and $\mathfrak{f}(k) = \mathcal{P}^{+}(k)\alpha^{+}(k)$

$$\begin{aligned}
\mathfrak{f}(k+1) &\overset{5.21}{=} \phi(k+1,k)\mathfrak{f}(k) + M(k+1)\alpha(k+1) \\
&\overset{6.25}{=} \phi(k+1,k)\mathcal{P}^{+}(k)\alpha^{+}(k) + M(k+1)\alpha(k+1) \\
&\overset{6.9}{=} \boxed{\left[\phi(k+1,k)\mathcal{P}^{+}(k)\phi^{*}(k+1,k) + M(k+1)\right]}\alpha(k+1)
\end{aligned}$$

$$\mathfrak{f}(k+1) = \mathcal{P}(k+1)\alpha(k+1)$$

Floppy decompositions summary



$$\mathcal{P}^+(k) = \overline{\tau}(k)\mathcal{P}(k)\overline{\tau}^*(k)$$

$$\mathcal{P}(k+1) = \phi(k+1,k)\mathcal{P}^+(k)\phi^*(k+1,k)$$

Defining $\psi(k+1,k)$

Define the ***articulated body transformation matrix***

$$\boxed{\psi(k+1,k) \triangleq \phi(k+1,k)\overline{\tau}(k)}$$

- $\psi(k+1,k)$ is a 6x6 matrix like $\phi(k+1,k)$
- However it is typically singular
- It depends on hinge properties
- Unlike $\phi(k+1,k)$ which propagates across rigid bodies, $\psi(k+1,k)$ propagates across articulated bodies

ATBI Riccati Equation

Claim:

Riccati equation

$$\boxed{\mathcal{P}(k+1) \overset{6.32}{=} \psi(k+1,k)\mathcal{P}(k)\psi^*(k+1,k) + M(k+1)}$$

Proof:

Have

$$\mathcal{P}(k+1) \triangleq \phi(k+1,k)\mathcal{P}^+(k)\phi^*(k+1,k) + M(k+1)$$

The result follows from substituting in

$$\mathcal{P}^+(k) \triangleq \mathcal{P}(k)\overline{\tau}^*(k)$$

and $\psi(k+1,k) \triangleq \phi(k+1,k)\overline{\tau}(k)$

Riccati vs Lyapunov Equations

Lyapunov equation for CRBs

$$\mathcal{R}(k) = \phi(k, k-1)\mathcal{R}(k-1)\phi^*(k, k-1) + M(k)$$

Riccati equation for ATBI

$$\mathcal{P}(k+1) \overset{6.32}{=} \psi(k+1, k)\mathcal{P}(k)\psi^*(k+1, k) + M(k+1)$$

They look similar, so why the different terminology?

Depends on $\mathcal{P}(k)$ and hence have **quadratic** terms

O(N) recursive gather algorithm for ATBIs

This is a tip to base gather recursion

```
$\mathcal{P}^{+}(0) = \mathbf{0}$
for $k$    $1 \cdots n$
    $\mathcal{P}(k) = \phi(k, k-1)\mathcal{P}^{+}(k-1)\phi^{*}(k, k-1) + M(k)$
    $\mathcal{D}(k) = H(k)\mathcal{P}(k)H^{*}(k)$
    $\mathcal{G}(k) = \mathcal{P}(k)H^{*}(k)\mathcal{D}^{-1}(k)$
    $\overline{\tau}(k) = \mathbf{I} - \mathcal{G}(k)H(k)$
    $\mathcal{P}^{+}(k) = \overline{\tau}(k)\mathcal{P}(k)$
end loop
```

Properties of the Articulated Body Inertia

P(k) is not a spatial inertia

The articulated body inertia P(k) acts like an inertia but is not a spatial inertia!

- It is a dense 6x6, symmetric, positive definite matrix

Comparison with $\mathcal{P}^{+}(\mathrm{k})$

$$\boxed{\mathcal{P}^{+}(\mathrm{k}) \triangleq \mathcal{P}(\mathrm{k})\overline{\tau}^{*}(\mathrm{k})}$$

- $\mathcal{P}^{+}(\mathrm{k})$ is symmetric, but <u>singular</u> and only positive semi-definite

- Moreover

$$\mathcal{P}(\mathrm{k}) \geqslant \mathcal{P}^{+}(\mathrm{k})$$

SHOW!

Comparison with other inertias

Also

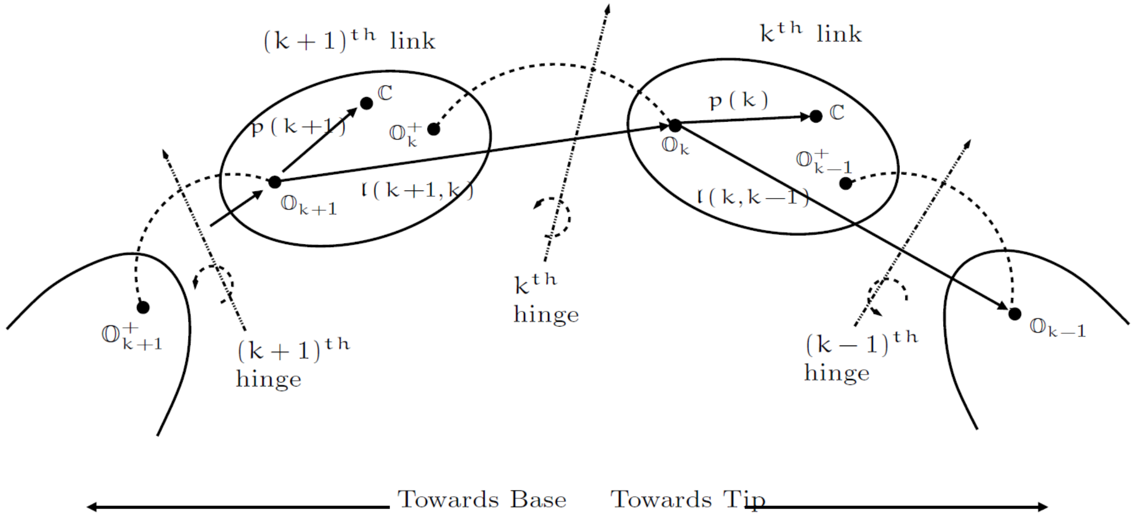
$$\mathcal{R}(k) \geqslant \mathcal{P}(k) \geqslant \mathbf{M}(k)$$

SHOW!

composite
body inertia

ATBI inertia for hinge special cases

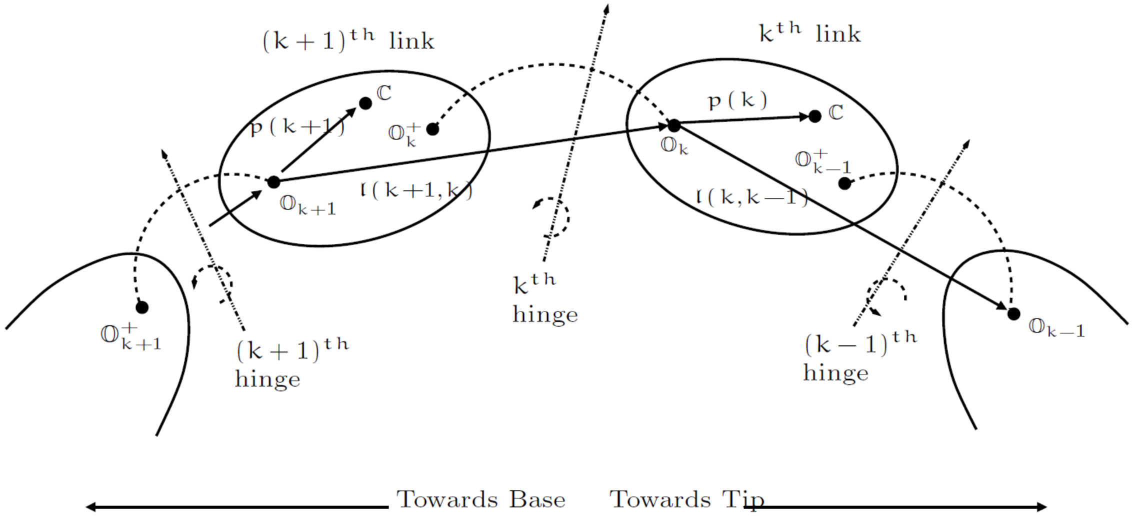
Special case: Locked hinge (0 dof)



Parent/child bodies rigidly coupled (no articulation)

- D(k) = 0,   G(k) = 0
- $\tau(k) = 0$, $\overline{\tau}(k) = I$
- $\mathcal{P}(k) = \mathcal{P}^{+}(k)$

Special case: Uncoupled hinge (6 dof)



Parent/child bodies uncoupled (no constraints)

- $D(k) = P(k) = G(k)$
- $\tau(k) = I$, $\overline{\tau}(k) = 0$
- $\mathcal{P}^+(k) = 0$

Non-Floppy Articulated body model

Allowing non-zero generalized forces

- For the floppy model, we assumed that the outboard generalized forces were zero and had

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k)$$

where $\mathcal{T}(j) = H(j)\mathfrak{f}(j) = \mathbf{0} \quad \forall\, j < k$

- What happens when the outboard generalized forces are <u>non-zero</u>?
- Look for decomposition of the form:

$$\boxed{\mathfrak{f}(k+1) = \mathcal{P}(k+1)\alpha(k+1) + \boxed{\mathfrak{z}(k+1)}}$$

residual to compensate for non-zero gen forces

Tip body’s ATBI decomposition

- Clearly know the answer for the tip body

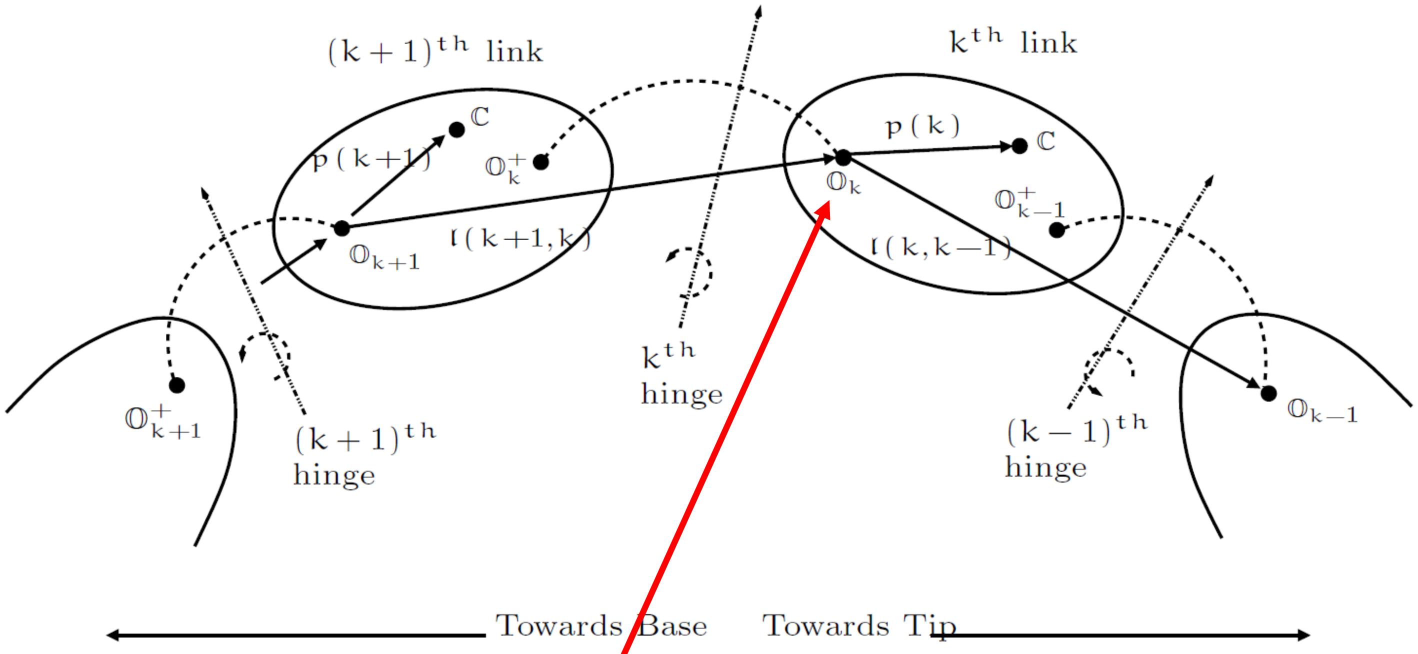
$$\boxed{\mathfrak{f}(1) = \mathcal{P}(1)\alpha(1)}$$

$$\boxed{\mathfrak{z}(1) = 0}$$

- Will use induction based argument to extend to other bodies
- So let us assume we have established the decomposition for the kth body

$$\boxed{\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)}$$

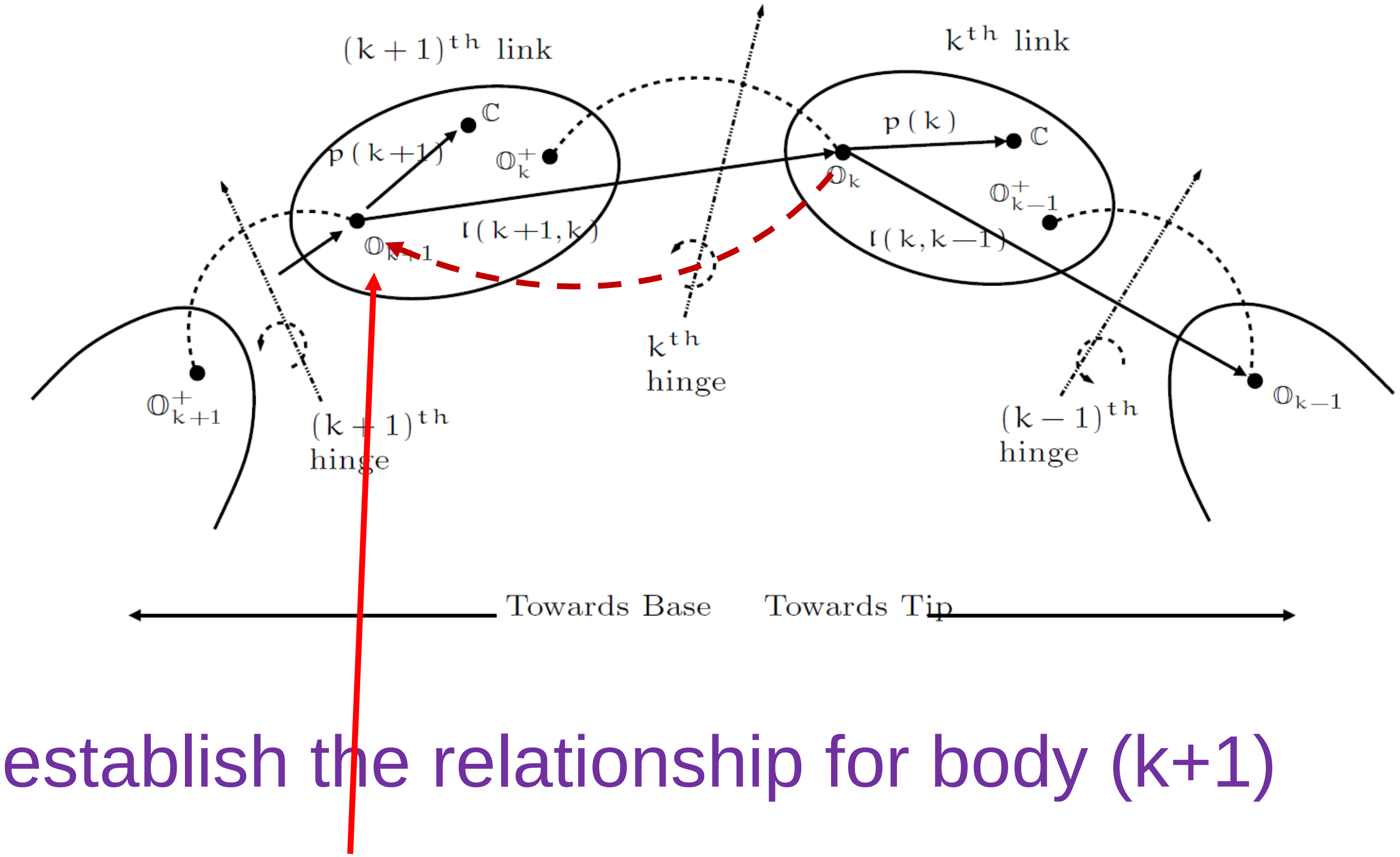
Induction based derivation - start



Assume we know the relationship for the kth body:

$$\boxed{\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)}$$

Induction based derivation - end



Want to establish the relationship for body (k+1)

$$\boxed{\mathfrak{f}(k+1) = \mathcal{P}(k+1)\alpha(k+1) + \mathfrak{z}(k+1)}$$

Moving to the inboard side of the hinge

Claim:

$$\boxed{\mathfrak{f}(k) = \mathcal{P}^{+}(k)\alpha^{+}(k) + \mathfrak{z}^{+}(k)}$$

where

$$\boxed{\mathfrak{z}^{+}(k) \overset{\triangle}{=} \overline{\tau}(k)\mathfrak{z}(k) + \mathcal{G}(k)\mathcal{T}(k)}$$

Proof:

$$\begin{aligned}
\mathfrak{f}(k) &\overset{6.16}{=} \overline{\tau}(k)\mathfrak{f}(k) + \tau(k)\mathfrak{f}(k) \overset{6.16,6.6}{=} \mathcal{G}(k)\mathrm{H}(k)\mathfrak{f}(k) + \overline{\tau}(k)\left[\mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)\right] \\
&\overset{5.21}{=} \mathcal{G}(k)\mathcal{T}(k) + \overline{\tau}(k)\mathcal{P}(k)\alpha(k) + \overline{\tau}(k)\mathfrak{z}(k) \\
&\overset{6.27,6.10}{=} \mathcal{G}(k)\mathcal{T}(k) + \mathcal{P}^{+}(k)\left[\alpha^{+}(k) + \mathrm{H}^{*}(k)\ddot{\theta}(k)\right] + \overline{\tau}(k)\mathfrak{z}(k) \\
&\overset{6.28}{=} \mathcal{G}(k)\mathcal{T}(k) + \boxed{\mathcal{P}^{+}(k)\alpha^{+}(k)} + \overline{\tau}(k)\mathfrak{z}(k)
\end{aligned}$$

Innovation term $\epsilon(k)$

Have

$$\mathfrak{z}^{+}(k) \triangleq \overline{\tau}(k)\mathfrak{z}(k) + \mathcal{G}(k)\mathcal{T}(k)$$

Define

$$\epsilon(k) \triangleq \mathcal{T}(k) - H(k)\mathfrak{z}(k)$$

innovation term

$$\nu(k) \triangleq \mathcal{D}^{-1}(k)\epsilon(k)$$

$$\mathfrak{z}^{+}(k) = \mathfrak{z}(k) + \mathcal{G}(k)\epsilon(k)$$

re-expression

Moving to (k+1) body frame

Claim: $\boxed{\mathfrak{f}(k+1) = \mathcal{P}(k+1)\alpha(k+1) + \mathfrak{z}(k+1)}$ where

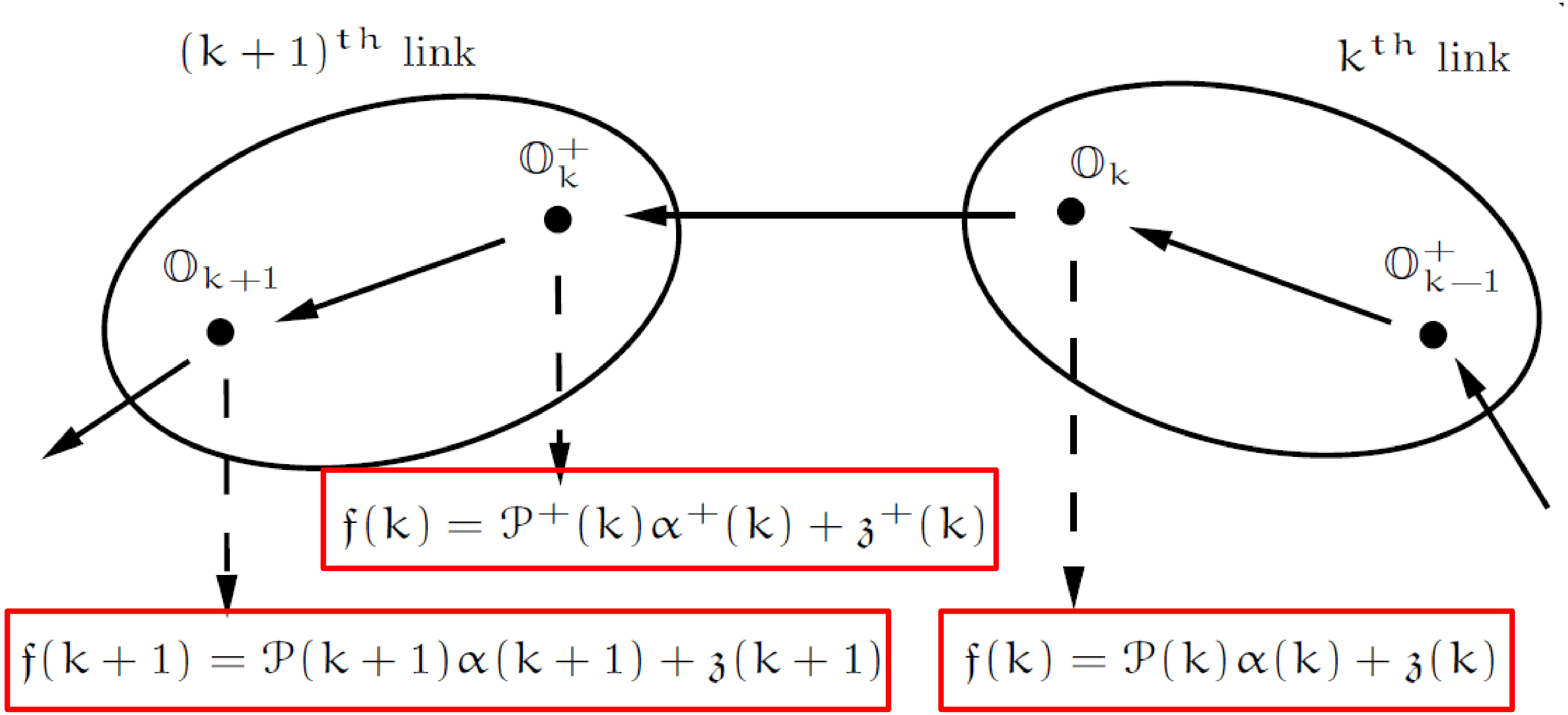
$$\boxed{\mathfrak{z}(k+1) \triangleq \phi(k+1)\mathfrak{z}^{+}(k) \overset{6.36,6.32}{=} \psi(k+1,k)\mathfrak{z}(k) + \mathcal{K}(k+1,k)\mathcal{T}(k)}$$

with $\mathcal{K}(k+1,k) \triangleq \phi(k+1,k)\mathcal{G}(k)$

Proof:

$$\begin{aligned}
\mathfrak{f}(k+1) &\overset{5.21}{=} \phi(k+1,k)\mathfrak{f}(k) + M(k+1)\alpha(k+1) \\
&\overset{6.35}{=} \phi(k+1,k)\left[\mathcal{P}^{+}(k)\alpha^{+}(k) + \mathfrak{z}^{+}(k)\right] + M(k+1)\alpha(k+1) \\
&\overset{6.9}{=} \boxed{\begin{aligned}&\phi(k+1,k)\mathcal{P}^{+}(k)\phi^{*}(k+1,k)\alpha(k+1) \\ &\quad + M(k+1)\alpha(k+1)\end{aligned}} + \phi(k+1,k)\mathfrak{z}^{+}(k) \\
&\overset{6.31}{=} \mathcal{P}(k+1)\alpha(k+1) + \phi(k+1,k)\mathfrak{z}^{+}(k)
\end{aligned}$$

ATBI decomposition summary



$$\mathfrak{f}(k) = \mathcal{P}^+(k)\alpha^+(k) + \mathfrak{z}^+(k)$$

$$\mathfrak{f}(k+1) = \mathcal{P}(k+1)\alpha(k+1) + \mathfrak{z}(k+1)$$

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)$$

O(N) recursive gather algorithm for the residual forces

Tip-to-base gather recursion

$$\begin{cases} \mathfrak{z}^{+}(0) = \mathbf{0} \\ \textbf{for } \boldsymbol{k} \quad \mathbf{1} \cdots \boldsymbol{n} \\ \quad \mathfrak{z}(k) = \phi(k, k-1)\mathfrak{z}^{+}(k-1) \\ \quad \epsilon(k) = \mathcal{T}(k) - H(k)\mathfrak{z}(k) \\ \quad \mathfrak{z}^{+}(k) = \mathfrak{z}(k) + \mathcal{G}(k)\epsilon(k) \\ \textbf{end loop} \end{cases}$$

This, together with earlier recursion for the ATBIs, is half the story for the O(N) forward dynamics algorithm!

Generalized accelerations

For the floppy case we had $\boxed{\ddot{\theta}(k) = -\mathcal{G}^*(k)\alpha^+(k)}$

Claim:

$$\ddot{\theta}(k) \overset{6.11}{=} \boxed{\nu(k)} - \mathcal{G}^*(k)\alpha^+(k)$$

non-floppiness
compensating term

Proof:

$$\begin{aligned}\ddot{\theta}(k) &\overset{6.43}{=} \mathcal{D}^{-1}(k)\{\epsilon(k) - H(k)\mathcal{P}(k)\phi^*(k+1,k)\alpha(k+1)\}\\ &\overset{6.44,6.13}{=} \nu(k) - \mathcal{G}^*(k)\phi^*(k+1,k)\alpha(k+1)\\ &\overset{6.39}{=} \nu(k) - \mathcal{K}^*(k+1,k)\alpha(k+1)\end{aligned}$$

Body spatial accelerations

Generally have

$$\boxed{\alpha(k) = \phi^*(k+1,k)\alpha(k+1) + H^*(k)\ddot{\theta}(k)}$$

Claim:

$$\alpha(k) \overset{6.32,6.39}{=} \boxed{\psi^*(k+1,k)}\alpha(k+1) + \boxed{H^*(k)\nu(k)}$$

hence articulated body transformation matrix

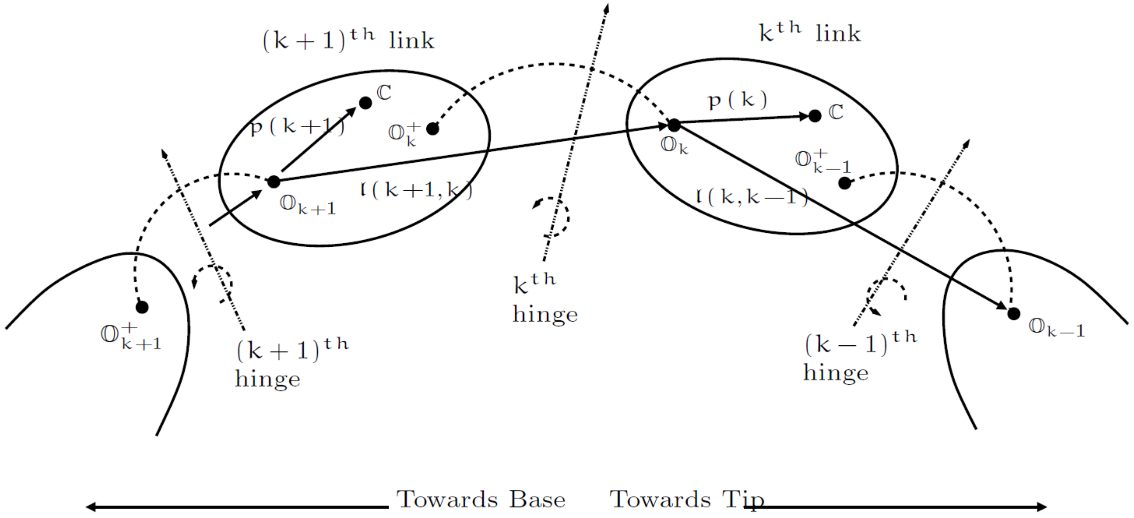
non-floppiness compensating term

Proof:

$$\begin{aligned}
\alpha(k) &\overset{5.21}{=} \phi^*(k+1,k)\alpha(k+1) + H^*(k)\ddot{\theta}(k) \\
&\overset{6.45}{=} \phi^*(k+1,k)\alpha(k+1) + H^*(k)\left[\nu(k) - \mathcal{K}^*(k+1,k)\alpha(k+1)\right] \\
&\overset{6.32,6.39}{=} \psi^*(k+1,k)\alpha(k+1) + H^*(k)\nu(k)
\end{aligned}$$

ATBI residual for hinge special cases

Special case: Locked hinge (0 dof)



Parent/child bodies rigidly coupled (no articulation)

- $D(k) = 0, \quad G(k) = 0$
- $\tau(k) = 0, \quad \overline{\tau}(k) = I$
- $\mathcal{P}(k) = \mathcal{P}^+(k)$

- $\ddot{\theta}(k) = \nu(k) = \epsilon(k) = 0$
- $\alpha^+(k) = \alpha(k)$
- $\mathfrak{z}^+(k) = \mathfrak{z}(k)$

Articulated body model

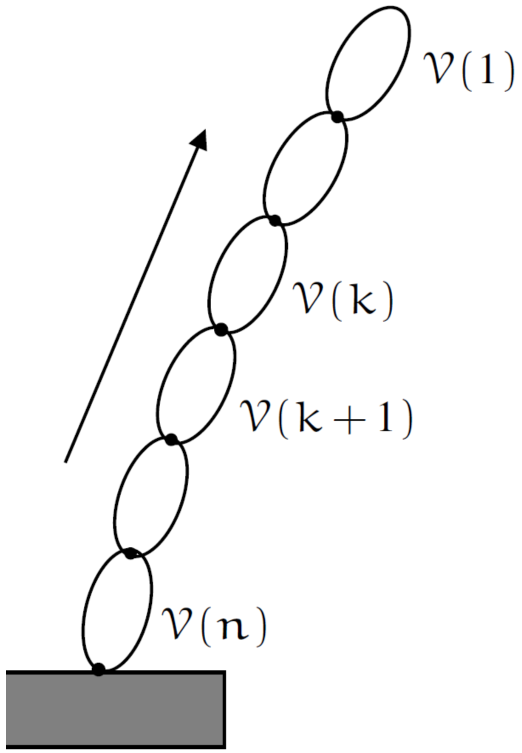
Articulated body model

- An improvement over terminal body model in not ignoring outboard bodies
- Residual term is zero if outboard generalized forces are zero, i.e. if outboard bodies are floppy

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k)$$

- The residual term accounts for the non-zero generalized forces of the outboard bodies
    - The gen forces of the inboard bodies does not matter

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \boxed{\mathfrak{z}(k)}$$



Inter-body spatial force decompositions

- Force decompositions consist of inertia + residual terms
- From the equations of motion we had

$$\mathfrak{f}(k) = M(k)\alpha(k) + \phi(k,k-1)\mathfrak{f}(k-1)$$

*depends on **kth** body*

*depends on **all** bodies*

- Using CRBs we have

$$\mathfrak{f}(k) = \mathcal{R}(k)\alpha(k) + \mathfrak{y}(k)$$

*depends on **outboard** bodies only*

*depends on **outboard** generalized accels*

- Using ATBI we have

$$\mathfrak{f}(k) = \mathcal{P}(k)\alpha(k) + \mathfrak{z}(k)$$

*depends on **outboard** bodies only*

*depends on **outboard** generalized forces*

Connections to Estimation Theory

The optimal estimation problem

Consider the noisy, discrete, time-domain dynamical system

state propagation matrix

state

$$x(k) = \phi(k, k-1)x(k-1) + w(k)$$

$$\mathcal{T}(k) = H(k)x(k)$$

output

time

white noise with covariance M(k)

Estimation problems

- **Optimal filtering:** At a given time k, and the **<u>past</u>** observations T(1) … T(k), determine the best estimate z(k) for the state x(k). This is a *<u>causal</u>* problem.

- **Optimal smoothing:** Given **<u>all</u>** observations – past and future, T(1) … T(n), determine the best estimate f(k) for x(k). This is an *<u>anti-causal</u>* problem.

Role of $\mathcal{E}_\phi$ and ϕ

$$\boxed{\begin{aligned} x(k) &= \phi(k, k-1)x(k-1) + w(k) \\ \mathcal{T}(k) &= H(k)x(k) \end{aligned}}$$

$x = \mathcal{E}_\phi x + w$

Re-expression of the time domain relationship

$x = \phi w$

Maping from the full input vector to the full state vector

$T = Hx$

Maping from the full state vector to the full output vector

Optimal Kalman filter

The optimal filtering process involves the following steps at each time instant:

1. Use a **Riccati equation** to propagate the estimation error covariance and define gains to use
2. Use the previous state estimate to **predict** the state at the current time
3. Extract **new information** (i.e. the innovations term) from the current observation
4. Use the innovations term to **update** the predict to compute the filter state estimate

Correspondence to dynamics

We will use same notation to show correspondence

- Estimation error covariance P(k)
- Riccati equation

Riccati equation

$$\boxed{\mathcal{P}(k+1) \overset{6.32}{=} \psi(k+1,k)\mathcal{P}(k)\psi^*(k+1,k) + M(k+1)}$$

- Predict step

predict term

$$\boxed{\mathfrak{z}(k+1) \triangleq \phi(k+1)\mathfrak{z}^+(k)}$$

- Update step

innovations term

$$\boxed{\epsilon(k) \triangleq \mathcal{T}(k) - H(k)\mathfrak{z}(k)}$$

optimal filter estimate

$$\boxed{\mathfrak{z}^+(k) = \mathfrak{z}(k) + \mathcal{G}(k)\epsilon(k)}$$

Optimal Kalman smoother

- Based on Bryson-Frazier method
- Uses a recursion going backwards in time
- Uses the stored optimal filter estimates
- Update the filter estimates using the backward recursion co-state to compute the smoothed state estimate

co-state

$$\alpha(k) \overset{2,6.39}{=} \psi^*(k+1,k)\alpha(k+1) + H^*(k)\nu(k)$$

optimal smoothed estimate

$$\boxed{\mathfrak{f}(k) = \mathfrak{z}(k) + \mathcal{P}(k)\alpha(k) = \mathfrak{z}^+(k) + \mathcal{P}^+(k)\alpha^+(k)}$$

Key covariance quantities

state covariance

$$\mathrm{cov}[x] = \phi M \phi^*$$

output covariance

$$\mathrm{cov}[T] = \mathcal{M}$$

filter estimate covariance

$$\begin{aligned}\mathrm{cov}[z] &= \phi K \mathcal{D}^{-1} K^* \phi^* = \tilde{\phi} \tau P \tau^* \tilde{\phi}^* \\ &= \phi M \phi^* - [P + \tilde{\phi} P + P \tilde{\phi}^*] \\ &= (R - P) + \tilde{\phi}(R - P) + (R - P)\tilde{\phi}^*\end{aligned}$$

innovations covariance

$$\mathrm{cov}[\epsilon] = D$$

innovations alt covariance

$$\mathrm{cov}[\nu] = \mathcal{D}^{-1}$$

filter error covariance

$$\mathrm{cov}[e_z] = \psi M \psi^* = \boxed{P} + \tilde{\psi} P + P \tilde{\psi}^*$$

Kalman gain

$$\boxed{\mathcal{G}(k) \triangleq \mathcal{P}(k) H^*(k) \mathcal{D}^{-1}(k)}$$

Summary

- Developed articulated body model for the decomposition of forces
  - Defined articulated body inertias and related quantities
  - Derived expression for residual forces
  - Developed O(N) gather algorithm for computing these quantities
- Described parallels with estimation theory

SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity