Dynamics and Real-Time Simulation (DARTS) Laboratory

Spatial Operator Algebra (SOA)

2. Single Rigid-Body Dynamics

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https://dartslab.jpl.nasa.gov/

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SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics
5. **Mass matrix -** composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
6. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
7. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
8. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

6D Spatial Notation Recap

Spatial notation offers concise & consistent transformation expressions for arbitrary non-CM points

rigid body transformation matrix

Spatial velocities $${}^{C}\mathcal{V}(A,C) = \phi^*(B,C)\ {}^{B}\mathcal{V}(A,B)$$

Spatial forces $${}^{B}\mathfrak{f}(B) = \phi(B,C)\ {}^{C}\mathfrak{f}(C)$$

Spatial inertia $$M(x) = \phi(x,y)M(y)\phi^*(x,y)$$

Kinetic energy $$\mathfrak{K}_e = \frac{1}{2}\mathcal{V}^*(x)M(x)\mathcal{V}(x) = \frac{1}{2}\mathcal{V}^*(y)M(y)\mathcal{V}(y)$$

Spatial momentum $$\mathfrak{h}(x) = \phi(x,y)\mathfrak{h}(y)$$

Spatial Notation Benefits

- Reduces number of equations by half
- Reduces number of terms in each equation by over half
- Reduces types of terms needed – the $\phi(x, y)$ *rigid body transformation matrix* does much of the work across the board
- Equations apply generally, not just CM
- See consistent patterns (repetition, duality) across the different transformations
- There are useful properties involving spatial cross product and $\phi(x, y)$

Single Rigid Body Dynamics

Outline

- Dynamical systems
- Choosing coordinates
- Single rigid body dynamics

Dynamical Systems for Multibody

Dynamical systems

General form of a continuous and smooth dynamical system:

state derivative → $\dot{x}(t)$, *state* → $x(t)$, *input* → $u(t)$, *output* → $y(t)$

$$\dot{x}(t) = g(x(t), u(t))$$ *state derivative equation*

$$y(t) = h(x(t), u(t))$$ *output equation*

Multibody dynamical systems

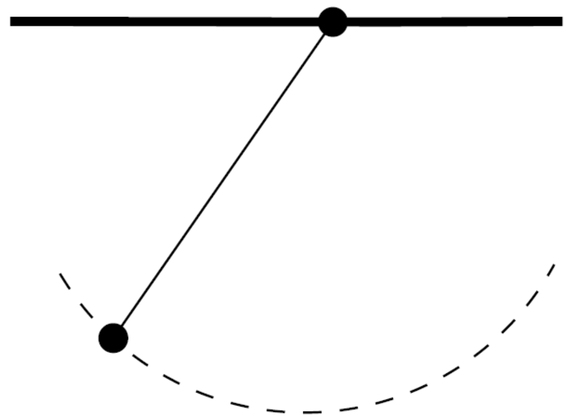
- Dynamical systems
  - **equations of motion** define the state derivative equation
  - **State x:** coordinates + velocities
  - **Inputs u:** external forces, gravity
  - **Outputs y:** are poses, velocities, loads etc

- Need to define the multibody coordinates
  - May be abstracted from physical space – but could be the same

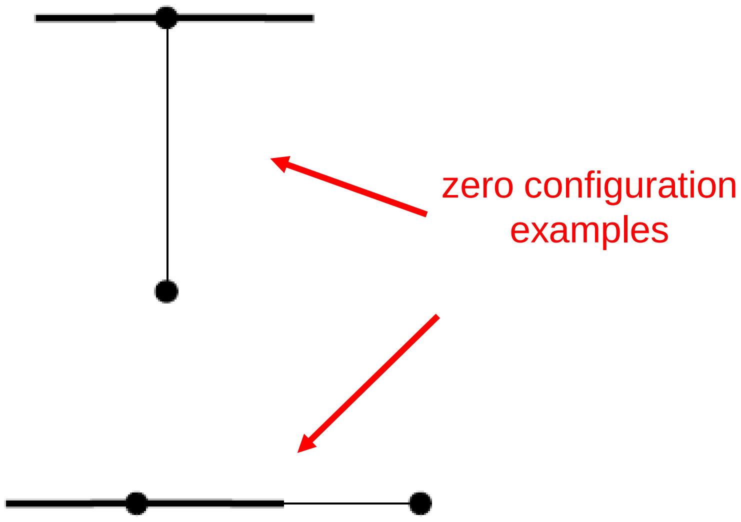
Generalized Coordinates

Multibody reference/zero-configuration

- The multibody zero-configuration defines the configuration where the state is zero
- Does not have to be a physically meaningful configuration
- Defines a reference configuration

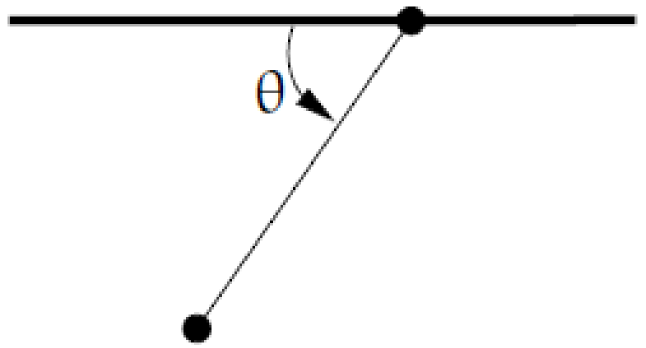


single link pendulum



Generalized coordinates θ

- **Generalized coordinates** θ specify the configuration of the system
- **Generalized velocities** are often (but not always) just the generalized coordinate rates $\dot{\theta}$
- The multibody dynamical **state** consists of generalized coordinates and generalized velocities

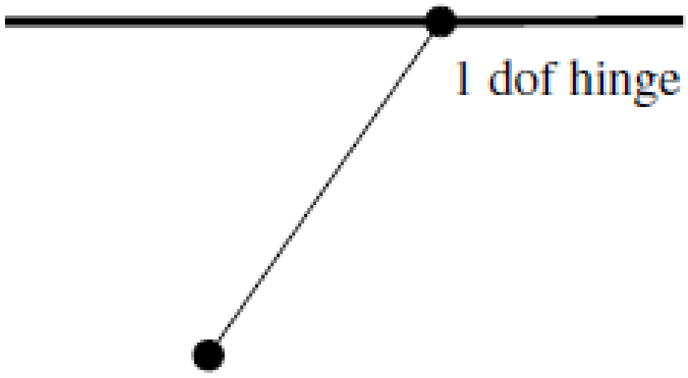


single link pendulum

Hinges and Constraints

Motion coordinates

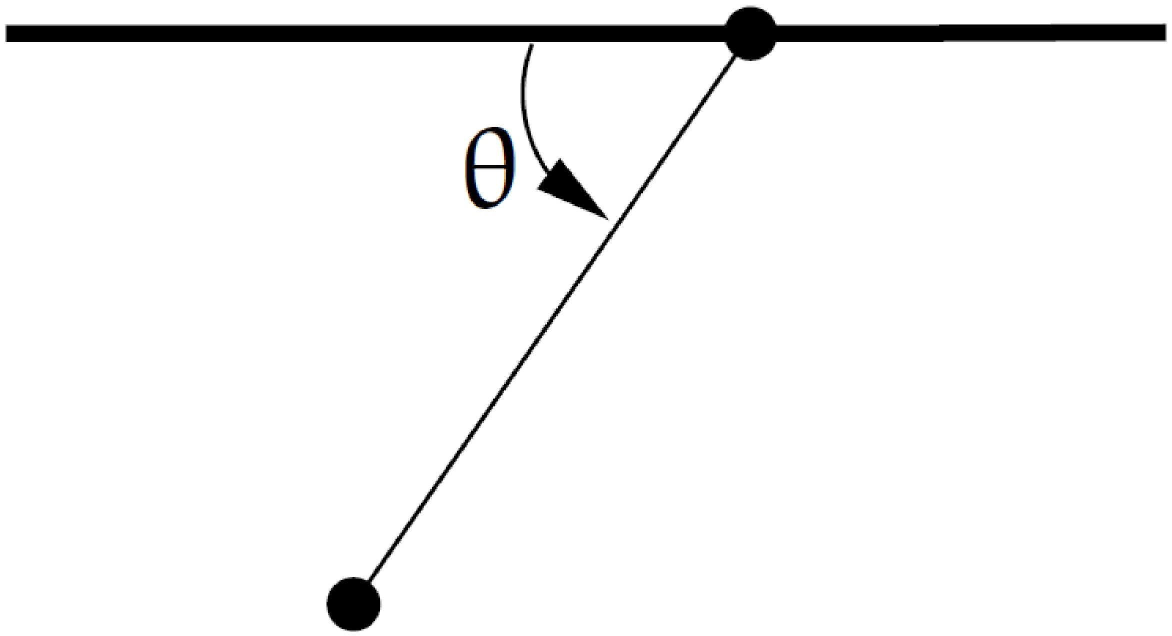
- Single link pendulum example
- Permissible motion can be described as
  - **Explicit:** a 1 degree of freedom (dof) **hinge**
  - **Implicit:** or equivalently as free 6 dof body, with 5 motion constraints



single link pendulum

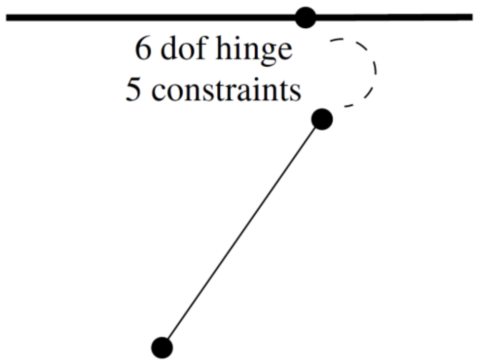
Hinges minimal coordinates approach

Explicit: a 1 degree of freedom (dof) **hinge**



Constraints approach

- **Implicit:** or equivalently as free 6 dof body, with 5 motion **constraints**

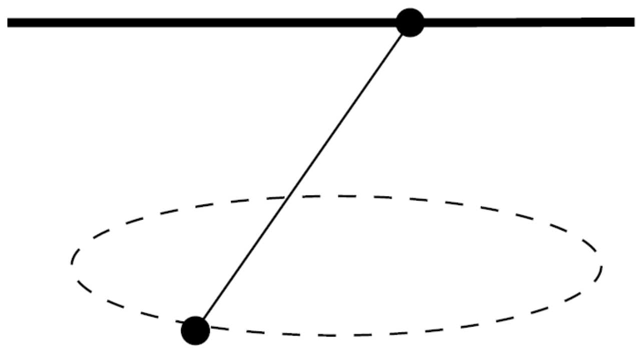


- **Alternative constraints approach: *Natural coordinates:*** more redundant coordinates using 2 points and a unit vector on the rigid body (9 dof, 8 motion constraints)
  - Avoids use of rotational coordinates

Multiple dof hinges

- We adopt the hinge **minimal coordinate approach**, and try to minimize use of constraints

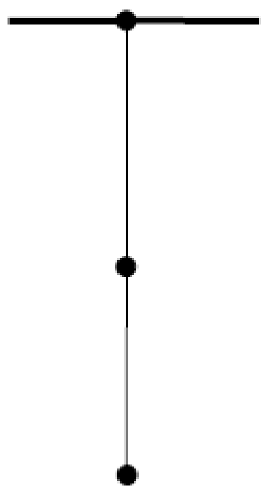
- A hinge can have more than 1 dof



Examples of Choosing Minimal Coordinates

Example: 2 link pendulum

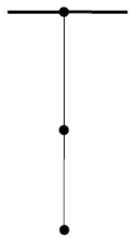
2 hinges and 2 dofs – overall motion is defined by the individual motion of the pair of hinges



zero configuration

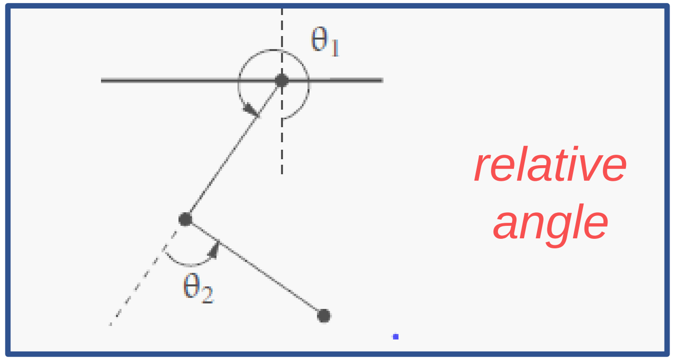
Generalized coordinates – 2 link pendulum

- What are the options for minimal coordinates?



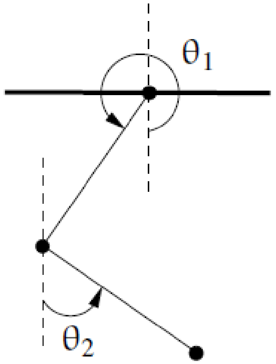
zero configuration

- <u>relative angle</u> generalized coordinate



*Choice of coordinates is **not unique***
Can easily go between these coordinate choices

- alternative <u>absolute angle</u> coordinates (deviation from the vertical) which is equally valid

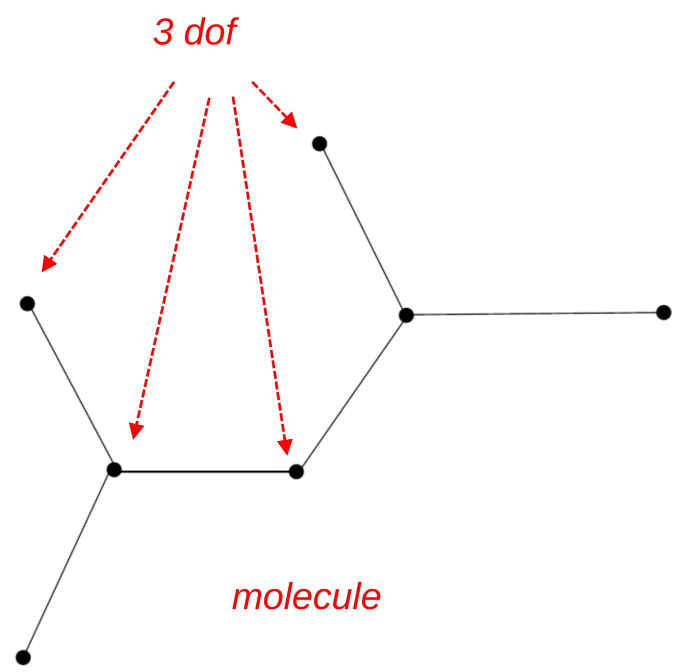


absolute angle

We will use **relative** coordinates

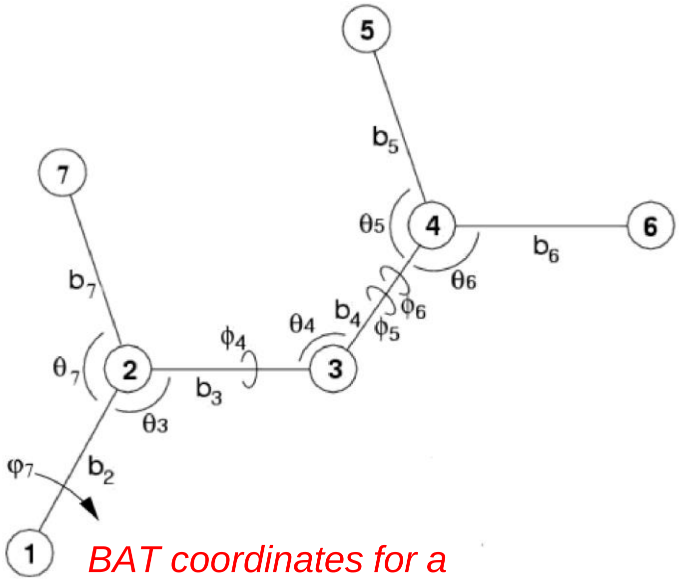
Example: Molecular dynamics models

- Collection of point mass atoms, motivates **3n dof** Cartesian coordinates – like for independent gas molecules
- However, with bonds, not all motion possible – bonds are stiff with little stretching and high frequency
- Common strategy is to freeze and eliminate the bond stretching dof to generate reduced order model and enable large time steps
  - This requires imposing constraints on the Cartesian coordinates



BAT coordinates for molecules

- Use alternative **bond/angle/torsion (BAT)** “internal” coordinates
- Same 3n number of dofs
- Coupled coordinates
- However these more naturally reflect the potentials and motion of a molecule – bond angle changes, torsional dofs
- Easy to eliminate stretching dofs by sampling ignoring these coordinates – no constraints required!
- In fact when doing entropy analysis, Cartesian motion are converted into the more appropriate BAT coordinates
    - Can avoid this by working with BAT coordinates in first place



BAT coordinates for a molecule

Choosing generalized coordinates

- The choice of coordinates is not unique
- Should be based on modeling needs
- Imposes requirements on algorithms

Generalized velocities

Generalized velocities

- Generalized velocities, denoted β parameterize the velocity motion space
- A common choice is to use the generalized coordinate rates $\dot{\theta}$ as generalized velocities

$$\beta = \dot{\theta}$$

- This is fine in many cases, but does not cover all situations
  - Lets look at holonomic and non-holonomic cases

Holonomic case

- Have a function of coordinates that describes the permissible motion

$$\mathfrak{d}(\theta, t) = \mathbf{0}$$

  - May be configuration and time dependent
  - For **pin** hinge: dx = dy = dz = eul(1) = eul(2) = 0

- Use gradient $G_c(\theta, t) \triangleq \nabla_\theta \mathfrak{d}(\theta, t)$ (velocity constraint matrix) to obtain the velocity relationship

$$\dot{\mathfrak{d}}(\theta, t) = G_c(\theta, t)\dot{\theta} - \mathfrak{U}(t) = \mathbf{0}$$

- Check **rank r** of gradient matrix, **dofs is (6-r).**
- Number of coordinate and velocity dofs is the <u>same</u>
- Orthogonal complement of gradient matrix specifies permitted relative spatial velocities across hinge

Non-holonomic case

- Start with similar velocity constraint equation

$$G_c(\theta, t)\dot{\theta} - \mathfrak{U}(t) = \mathbf{0}$$

- However, in non-holonomic case, the velocity constraint matrix may **not be a gradient**
- *Fewer velocity dofs can change configuration over larger dimensional coordinate space!*

- **Examples**
  - Ball rolling on the ground: only 3 velocity dofs (all rotational), but can change 5 coordinates - all but z value
  - Car: only 2 velocity dofs, but can parallel park, i.e. change 3 coordinates – x, y and heading

Non-holonomic case (contd)

- Velocity constraint equation

$$G_c(\theta, t)\dot{\theta} - \mathfrak{U}(t) = \mathbf{0}$$

- In non-holonomic case, the velocity constraint matrix may **not be a gradient**
- Check rank r of gradient matrix, **dofs is (6-r).**
- Coordinate and velocity dofs may **not** be the same
- Orthogonal complement of the gradient matrix specifies permitted hinge motion (i.e. relative hinge spatial velocities)

Degrees of freedom for a hinge usually refer to the velocity space dofs

Quasi-velocities

- Take the case of a tumbling rigid body – has position and attitude dofs – 3 each
- With Euler angle rates, etc for generalized velocities, the attitude dynamics are complicated
- Dynamics much simpler using angular velocities

$$\mathscr{J}\,\dot{\omega} + \tilde{\omega}\,\mathscr{J}\,\omega = \mathfrak{f}$$

  - So why not use angular velocities as generalized velocities!
  - Angular velocities are however **not integrable** (i.e. not time derivatives). Not a problem.
- Such non-integrable generalized velocity coordinates are called **quasi-velocities**

Generalized forces $\mathcal{T}$

- For a rigid body power is given by

$$\text{power} = \mathfrak{f}^* \cdot \mathcal{V}$$

- Say $\mathcal{V} = A\beta$, thus

$$\text{power} = \mathfrak{f}^* \cdot A\beta = \mathcal{T}^* \beta, \text{ with } \mathcal{T} = A^* \mathfrak{f}$$

- We have transformed power relationship from physical to generalized coordinates domain, and used it to define the **generalized force** $\mathcal{T}$.
- Given a choice for generalized velocities β, the power relationship automatically defines what $\mathcal{T}$ should be

Example: Transforming generalized velocities

- Lets say we have a different choice for generalized velocities

$$\beta_1 = A(\theta)\beta$$

- Then by power relationship

$$\text{power} = \mathcal{T}^* \cdot \beta = \mathcal{T}^* \cdot A^{-1}\beta_1 = \mathcal{T}_1^* \cdot \beta_1 \quad \text{where} \quad \mathcal{T}_1 = A^{-*}\mathcal{T}$$

- So the generalized velocities and forces go together – compatible pairs defined by the power relationship
- Transforming the generalized velocities, transforms the generalized forces as well
- Examples
  - Say doubling
  - Picking different units
  - Picking combinations – relative to absolute angle rates

Multibody state and state derivatives

- State: $x = (\theta, \beta)$
- State derivative: $\dot{x} = (\dot{\theta}, \dot{\beta})$
- Next look at equations of motion for single rigid body

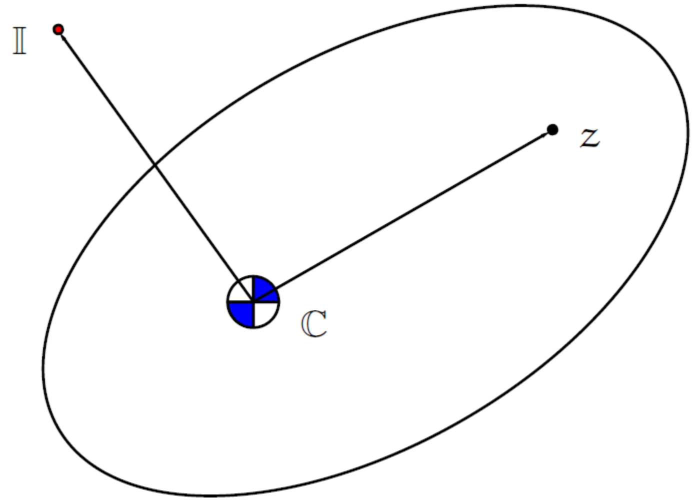
$$\dot{x}(t) = g(x(t), u(t))$$

Single Rigid Body

Equations of motion

- More than the generalized coordinates, the choice of generalized velocities directly effects the form of the equations of motion

- We now look at the equations of motion of a single rigid body for different choices of generalized velocities



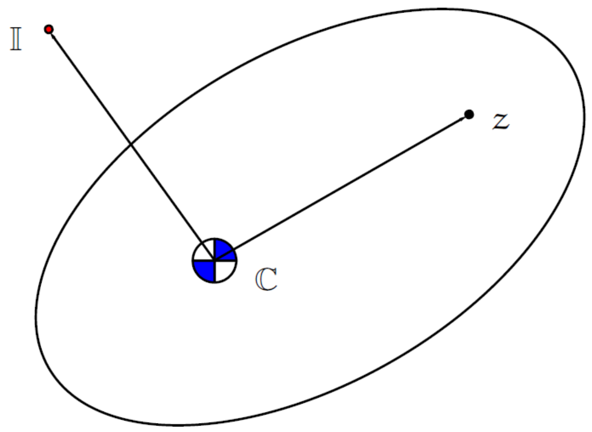
Single rigid body generalized velocities

Generalized velocity coordinate options

$$\beta_{\mathcal{J}} = {}^{\mathbb{I}}\mathcal{V}(\mathbb{C})$$ sp. velocity of CM, inertial frame

$$\boldsymbol{\beta}_{\mathcal{J}} = {}^{\mathbb{I}}\boldsymbol{\mathcal{V}}(\boldsymbol{z})$$ sp. velocity of z, inertial frame

$$\beta_{\mathbb{B}} = {}^{\mathbb{B}}\mathcal{V}(z)$$ sp. velocity of z, body frame



$$\beta_{\mathbb{I}} \triangleq {}^{\mathbb{I}}\mathcal{V}_{\mathbb{I}} \triangleq \phi^*(\mathbb{C}, \mathbb{I})\mathcal{V}(\mathbb{C})$$ inertial reference point I sp. vel

All of these options include angular velocity coordinates, and are hence quasi-velocities.

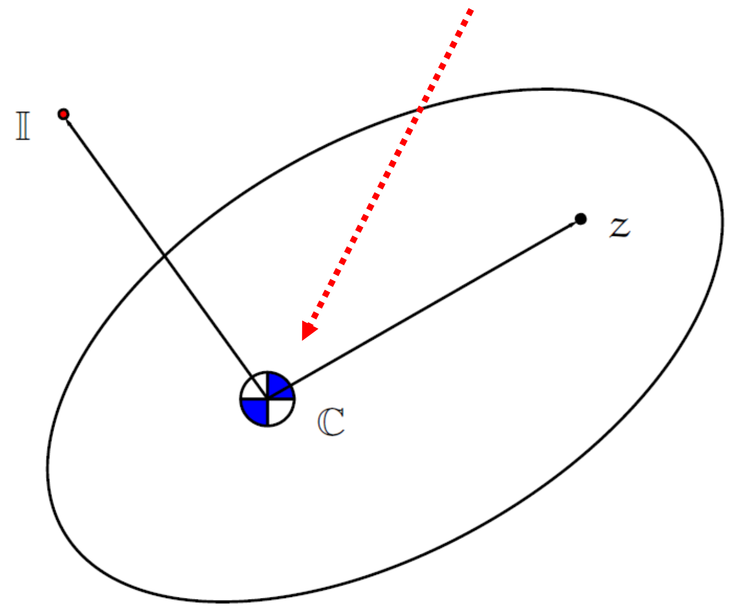
Single rigid body

Center of Mass Dynamics

Generalized velocities – center of mass, inertial derivative

Coordinates of the spatial velocity of the center of mass (CM) in inertial frame

$$\beta_{\mathcal{J}} = {}^{\mathbb{I}}\mathcal{V}(\mathbb{C})$$



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Center of mass dynamics

- Derivative of linear and angular momenta at 3D component level

angular — linear

$$\mathrm{N}(\mathbb{C}) = \frac{\mathrm{d}_{\mathbb{I}} \mathcal{J}(\mathbb{C})\omega(\mathbb{C})}{\mathrm{dt}} \quad \text{and} \quad \mathrm{F}(\mathbb{C}) = \frac{\mathrm{d}_{\mathbb{I}} m v(\mathbb{C})}{\mathrm{dt}}$$

- Equivalently using spatial notation

spatial

$$\mathfrak{f}(\mathbb{C}) = \frac{\mathrm{d}_{\mathbb{I}} M(\mathbb{C}) \mathcal{V}(\mathbb{C})}{\mathrm{dt}} = \frac{\mathrm{d}_{\mathbb{I}} \mathfrak{h}(\mathbb{C})}{\mathrm{dt}}$$

- Spatial momentum is conserved in the absence of external spatial forces

Some notation

Component forms of spatial vectors

$$\mathcal{X} = \mathcal{X}^{\omega} + \mathcal{X}^{v}$$

decomposition

$$\mathcal{V}(x) \triangleq \begin{bmatrix} \omega(x) \\ v(x) \end{bmatrix}$$

$$\mathcal{V}^{\omega}(x) \triangleq \begin{bmatrix} \omega(x) \\ \mathbf{0} \end{bmatrix}$$

angular component

$$\mathcal{V}^{v}(x) \triangleq \begin{bmatrix} \mathbf{0} \\ v(x) \end{bmatrix}$$

linear component

$$\tilde{\mathcal{V}}^{\omega}(z) = \begin{pmatrix} \tilde{\omega} & \mathbf{0} \\ \mathbf{0} & \tilde{\omega} \end{pmatrix}$$

Time derivative of spatial inertia

- Inertial and body frame spatial inertia relationship

$$^{\mathbb{I}}\mathcal{M}(z) = \begin{pmatrix} ^{\mathbb{I}}\mathfrak{R}_{\mathbb{B}} & \mathbf{0} \\ \mathbf{0} & ^{\mathbb{I}}\mathfrak{R}_{\mathbb{B}} \end{pmatrix} {}^{\mathbb{B}}\mathcal{M}(z) \begin{pmatrix} ^{\mathbb{B}}\mathfrak{R}_{\mathbb{I}} & \mathbf{0} \\ \mathbf{0} & ^{\mathbb{B}}\mathfrak{R}_{\mathbb{I}} \end{pmatrix}$$

- Time derivative of the inertial spatial inertia

$$\dot{\mathcal{M}}(z) = \begin{pmatrix} \widetilde{\omega} & \mathbf{0} \\ \mathbf{0} & \widetilde{\omega} \end{pmatrix} {}^{\mathbb{B}}\mathcal{M}(z) - \mathcal{M}(z) \begin{pmatrix} \widetilde{\omega} & \mathbf{0} \\ \mathbf{0} & \widetilde{\omega} \end{pmatrix}$$

$$= \widetilde{\mathcal{V}}^{\omega}(z)\mathcal{M}(z) - \mathcal{M}(z)\widetilde{\mathcal{V}}^{\omega}(z)$$

CM equations of motion

The CM equations of motion are

$$\mathfrak{f}(\mathbb{C}) = \mathcal{M}(\mathbb{C})\,\dot{\boldsymbol{\beta}}_{\mathcal{J}}(\mathbb{C}) + \mathfrak{b}_{\mathcal{J}}(\mathbb{C})$$

sp. force at CM — sp. inertia at CM — gen. accel — gyroscopic

with gyroscopic term

familiar component level term

$$\mathfrak{b}_{\mathcal{J}}(\mathbb{C}) \triangleq \tilde{\mathcal{V}}^{\omega}(\mathbb{C})\mathcal{M}(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C})$$

$$= \overline{\mathcal{V}}^{\omega}(\mathbb{C})\mathcal{M}(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C}) = \begin{bmatrix} \tilde{\omega}\,\mathcal{J}(\mathbb{C})\,\omega \\ \mathbf{0} \end{bmatrix}$$

CM equations of motion (derivation)

Have

$$\mathfrak{f}(\mathbb{C}) = \frac{d_{\mathbb{I}} M(\mathbb{C}) \mathcal{V}(\mathbb{C})}{dt}$$

← *spatial momentum*

$$\begin{aligned}
\mathfrak{f}(\mathbb{C}) &\overset{2.21}{=} M(\mathbb{C})\dot{\boldsymbol{\beta}}_{\mathfrak{J}}(\mathbb{C}) + \frac{d_{\mathbb{I}} M(\mathbb{C})}{dt}\mathcal{V}(\mathbb{C}) \\
&\overset{2.22}{=} M(\mathbb{C})\dot{\boldsymbol{\beta}}_{\mathfrak{J}}(\mathbb{C}) + \underbrace{\left[\widetilde{\mathcal{V}}^{\omega}(\mathbb{C})M(\mathbb{C}) - M(\mathbb{C})\,\widetilde{\mathcal{V}}^{\omega}(\mathbb{C})\right]}_{\dot{M}}\mathcal{V}(\mathbb{C}) \\
&\overset{1.25}{=} M(\mathbb{C})\dot{\boldsymbol{\beta}}_{\mathfrak{J}}(\mathbb{C}) + \widetilde{\mathcal{V}}^{\omega}(\mathbb{C})M(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C}) \\
&\qquad + \left[\widetilde{\mathcal{V}}^{\omega}(\mathbb{C})M(\mathbb{C})\mathcal{V}^{\nu}(\mathbb{C}) - M(\mathbb{C})\,\widetilde{\mathcal{V}}^{\omega}(\mathbb{C})\mathcal{V}^{\nu}(\mathbb{C})\right] \\
&\overset{2.10}{=} M(\mathbb{C})\dot{\boldsymbol{\beta}}_{\mathfrak{J}}(\mathbb{C}) + \widetilde{\mathcal{V}}^{\omega}(\mathbb{C})M(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C})
\end{aligned}$$

CM gyroscopic term

$$\mathfrak{b}_{\mathfrak{J}}(\mathbb{C}) \triangleq \widetilde{\mathcal{V}}^{\omega}(\mathbb{C})\mathrm{M}(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C})$$

$$= \overline{\mathcal{V}}^{\omega}(\mathbb{C})\mathrm{M}(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C}) = \begin{bmatrix} \tilde{\omega}\mathscr{J}(\mathbb{C})\omega \\ \mathbf{0} \end{bmatrix}$$

The gyroscopic term in the CM equations of motion does no work, i.e.

$$\mathcal{V}^*(\mathbb{C})\mathfrak{b}_{\mathfrak{J}}(\mathbb{C}) = \mathbf{0}$$

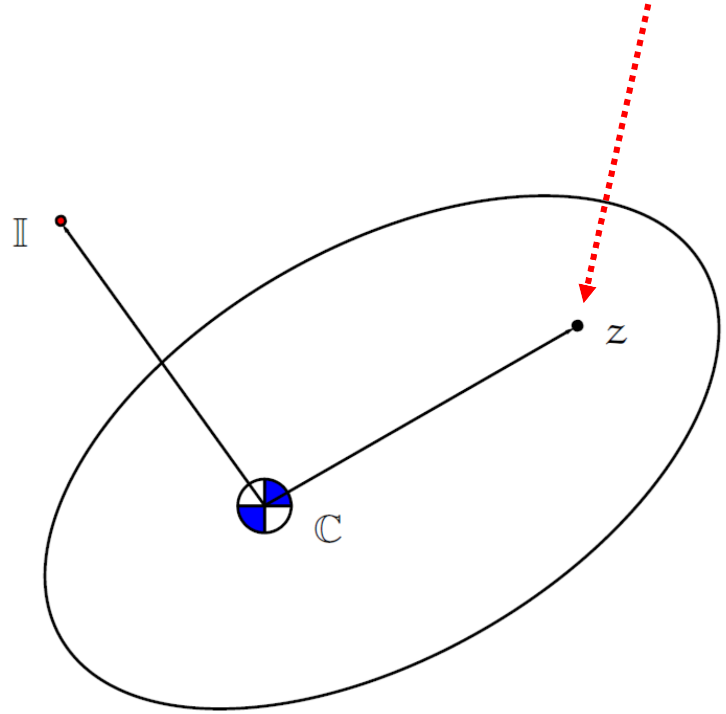
Single rigid body

General point dynamics
Inertial derivatives

Generalized velocities – arbitrary point, inertial derivative

Coordinates of spatial velocity of z in inertial frame

$$\beta_{\mathcal{J}} = {}^{\mathbb{I}}\mathcal{V}(z)$$



Generalized velocities – arbitrary point, inertial derivative

- Generalized velocity – spatial velocity in inertial frame

$$\beta_{\mathcal{J}} = {}^{\mathbb{I}}\mathcal{V}(z)$$

- General acceleration relationship for $\mathcal{V}(y) = \phi^*(x,y)\,\mathcal{V}(x)$

$$\dot{\beta}_{\mathcal{J}}(y) \triangleq \frac{d_{\mathbb{I}}\mathcal{V}(y)}{dt} = \phi^*(x,y)\dot{\beta}_{\mathcal{J}}(x) - \widetilde{\mathcal{V}}(y)\mathcal{V}(x)$$

$$= \phi^*(x,y)\dot{\beta}_{\mathcal{J}}(x) + \begin{bmatrix} \mathbf{0} \\ \tilde{\omega}\,\tilde{\omega}\,\mathfrak{l}(x,y) \end{bmatrix}$$

Coriolis term

- Relationship to CM generalized velocities

$$\dot{\beta}_{\mathcal{J}}(z) \triangleq \frac{d_{\mathbb{I}}\mathcal{V}(z)}{dt} = \phi^*(\mathbb{C},z)\dot{\beta}_{\mathcal{J}}(\mathbb{C}) + \begin{bmatrix} \mathbf{0} \\ -\tilde{\omega}\,\tilde{\omega}\,p(z) \end{bmatrix}$$

Equations of Motion with $\beta_{\mathcal{J}} = {}^{\mathbb{I}}\mathcal{V}(z)$

The equations of motion with coordinates of generalized velocities of point z in inertial frame

$$\mathfrak{f}(z) = M(z)\dot{\beta}_{\mathcal{J}}(z) + \mathfrak{b}_{\mathcal{J}}(z)$$

with gyroscopic term

$$\mathfrak{b}_{\mathcal{J}}(z) \triangleq \widetilde{\mathcal{V}}^{\omega}(z)M(z)\mathcal{V}^{\omega}(z) = \overline{\mathcal{V}}^{\omega}(z)M(z)\mathcal{V}^{\omega}(z)$$

$$= \begin{bmatrix} \widetilde{\omega}\,\mathcal{J}(z)\omega \\ \mathfrak{m}\,\widetilde{\omega}\,\widetilde{\omega}\,\mathfrak{p}(z) \end{bmatrix}$$

Derivation of equations of motion

Uses CM equations of motion

$$
\begin{aligned}
\mathcal{M}(z)\dot{\boldsymbol{\beta}}_{\mathcal{J}}(z) &\overset{2.25}{=} \mathcal{M}(z)\left(\phi^*(\mathbb{C},z)\dot{\boldsymbol{\beta}}_{\mathcal{J}}(\mathbb{C}) + \begin{bmatrix} \mathbf{0} \\ -\tilde{\omega}\,\tilde{\omega}\,p(z) \end{bmatrix}\right) \\
&\overset{2.12,2.7}{=} \phi(z,\mathbb{C})\mathcal{M}(\mathbb{C})\dot{\boldsymbol{\beta}}_{\mathcal{J}}(\mathbb{C}) - \begin{bmatrix} m\,\tilde{p}(z)\,\tilde{\omega}\,\tilde{\omega}\,p(z) \\ m\,\tilde{\omega}\,\tilde{\omega}\,p(z) \end{bmatrix} \\
&\overset{2.23,A.1}{=} \phi(z,\mathbb{C})\left(\mathfrak{f}(\mathbb{C}) - \begin{bmatrix} \tilde{\omega}\,\mathscr{J}(\mathbb{C})\omega \\ \mathbf{0} \end{bmatrix}\right) - \begin{bmatrix} -m\,\tilde{\omega}\,\tilde{p}(z)\,\tilde{p}(z)\omega \\ m\,\tilde{\omega}\,\tilde{\omega}\,p(z) \end{bmatrix} \\
&\overset{1.66}{=} \mathfrak{f}(z) - \begin{bmatrix} \tilde{\omega}\left(\mathscr{J}(\mathbb{C}) - m\,\tilde{p}(z)\,\tilde{p}(z)\right)\omega \\ m\,\tilde{\omega}\,\tilde{\omega}\,p(z) \end{bmatrix} \\
&\overset{2.11}{=} \mathfrak{f}(z) - \begin{bmatrix} \tilde{\omega}\,\mathscr{J}(z)\omega \\ m\,\tilde{\omega}\,\tilde{\omega}\,p(z) \end{bmatrix}
\end{aligned}
$$

Gyroscopic term with $\beta_{\mathfrak{J}} = {}^{\mathbb{I}}\mathcal{V}(z)$

- Unlike at CM, the gyroscopic forces do work
- Moreover, the spatial momentum about z is not constant in the inertial frame in the absence of external forces!

$$\mathfrak{f}(z) \neq \frac{d_{\mathbb{I}}\mathfrak{h}(z)}{dt}$$

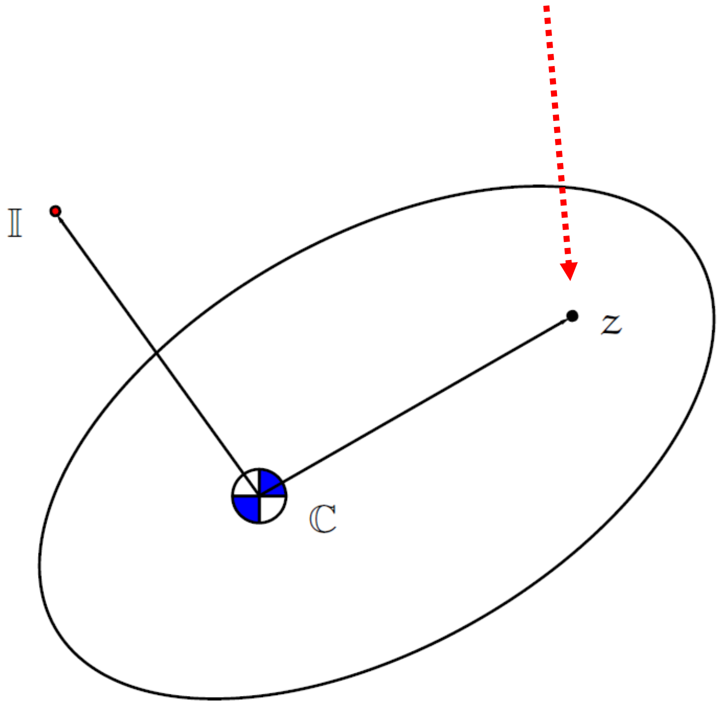
Single rigid body

General point dynamics
Body derivatives

Generalized velocities – arbitrary point, body derivative

Coordinates of spatial velocity of z in body frame

$$\beta_{\mathbb{B}} = {}^{\mathbb{B}}\mathcal{V}(z)$$



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Generalized velocity relationship

Relationship between body frame and inertial frame generalized velocities

$$\beta_{\mathbb{B}} = {}^{\mathbb{B}}\mathcal{V}(z) \qquad \beta_{\mathcal{I}} = {}^{\mathbb{I}}\mathcal{V}(z)$$

$$\dot{\beta}_{\mathbb{B}}(z) = \dot{\beta}_{\mathcal{I}}(z) - \begin{bmatrix} 0 \\ \tilde{\omega}\, v(z) \end{bmatrix}$$

Equations of Motion with $\beta_{\mathbb{B}} = {}^{\mathbb{B}}\mathcal{V}(z)$

The equations of motion with coordinates of generalized velocities of point z in body frame

$$\mathfrak{f}(z) = M(z)\dot{\beta}_{\mathbb{B}}(z) + \mathfrak{b}(z)$$

with gyroscopic term

$$\mathfrak{b}(z) \triangleq \overline{\mathcal{V}}(z)\mathfrak{h}(z) = \overline{\mathcal{V}}(z)M(z)\mathcal{V}(z)$$

Derivation of equations of motion

$$
\begin{aligned}
\mathcal{M}(z)\dot{\beta}_{\mathbb{B}}(z) &\overset{2.19}{=} \mathcal{M}(z)\left(\dot{\beta}_{\mathfrak{J}}(z) - \begin{bmatrix} 0 \\ \tilde{\omega}\, v(z) \end{bmatrix}\right) \\
&\overset{2.26,2.7}{=} \mathfrak{f}(z) - \begin{bmatrix} \tilde{\omega}\mathcal{J}(z)\omega \\ m\,\tilde{\omega}\,\tilde{\omega}\; p(z) \end{bmatrix} - \begin{bmatrix} m\,\tilde{p}(z)\,\tilde{\omega}\; v(z) \\ m\,\tilde{\omega}\; v(z) \end{bmatrix} \\
&= \mathfrak{f}(z) - \begin{bmatrix} \tilde{\omega}\mathcal{J}(z)\omega + m\,\tilde{p}(z)\,\tilde{\omega}\; v(z) \\ m\,\tilde{\omega}\,\tilde{\omega}\; p(z) + m\,\tilde{\omega}\; v(z) \end{bmatrix} \\
&\overset{A.1}{=} \mathfrak{f}(z) - \begin{bmatrix} \tilde{\omega}\mathcal{J}(z)\omega - m\left(\tilde{\omega}\,\tilde{v}(z)p(z) + \tilde{v}(z)\,\tilde{p}(z)\omega\right) \\ m\,\tilde{\omega}\left(\tilde{\omega}\; p(z) + v(z)\right) \end{bmatrix} \\
&= \mathfrak{f}(z) - \begin{bmatrix} \tilde{\omega}\left(\mathcal{J}(z)\omega + m\,\tilde{p}(z)v(z)\right) - m\,\tilde{v}(z)\,\tilde{p}(z)\omega \\ m\,\tilde{\omega}\left(\tilde{\omega}\; p(z) + v(z)\right) \end{bmatrix} \\
&= \mathfrak{f}(z) - \overline{\mathcal{V}}(z)\begin{bmatrix} \mathcal{J}(z)\omega + m\,\tilde{p}(z)v(z) \\ -m\,\tilde{p}(z)\omega + mv(z) \end{bmatrix} \\
&\overset{2.7}{=} \mathfrak{f}(z) - \overline{\mathcal{V}}(z)\mathcal{M}(z)\mathcal{V}(z) \overset{2.16}{=} \mathfrak{f}(z) - \overline{\mathcal{V}}(z)\mathfrak{h}(z)
\end{aligned}
$$

Gyroscopic term with $\beta_{\mathbb{B}} = {}^{\mathbb{B}}\mathcal{V}(z)$

- Once again, the gyroscopic spatial force does no work

$$\mathcal{V}^*(z)\mathfrak{b}(z) \overset{2.28}{=} \mathcal{V}^*(z)\,\overline{\mathcal{V}}(z)\mathfrak{h}(z) \overset{1.27}{=} 0$$

- Kinetic energy conservation easy to verify

$$\frac{1}{2}\frac{\mathrm{d}\mathcal{V}^*(z)\mathrm{M}(z)\mathcal{V}(z)}{\mathrm{d}t} = \mathcal{V}^*(z)\mathrm{M}(z)\dot{\boldsymbol{\beta}}_{\mathbb{B}}(z) \overset{2.28}{=} -\mathcal{V}^*(z)\mathfrak{b}(z) \overset{2.29}{=} 0$$

- Spatial momentum about z not conserved

Single rigid body

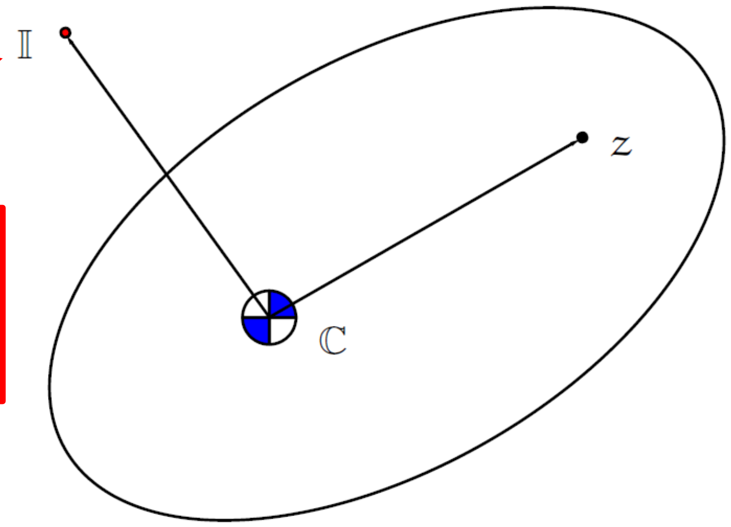
Inertial Reference Point dynamics

Generalized velocities – inertially referenced spatial velocity

Coordinates of inertially referenced spatial velocity

- Spatial velocity of **I**, as if the frame were rigidly attached to the body

$$\beta_{\mathbb{I}} \triangleq {}^{\mathbb{I}}\mathcal{V}_{\mathbb{I}} \triangleq \phi^*(\mathbb{C}, \mathbb{I})\mathcal{V}(\mathbb{C})$$



Inertially referenced spatial velocity

The inertially referenced spatial velocity is the same for all points x on the rigid body!

$$\mathcal{V}_{\mathbb{I}} = \phi^*(x, \mathbb{I})\ \mathcal{V}(x)$$

does not depend on the choice of x

Inertially referenced spatial Inertia

Inertially referenced spatial inertia

$$\mathbf{M}_{\mathbb{I}} \triangleq \phi(\mathbb{I},\mathbb{C})\mathbf{M}(\mathbb{C})\phi^*(\mathbb{I},\mathbb{C}) = \begin{pmatrix} \mathscr{J}_{\mathbb{I}} & \mathfrak{m}\,\widetilde{\mathbf{p}}_{\mathbb{I}} \\ -\mathfrak{m}\,\widetilde{\mathbf{p}}_{\mathbb{I}} & \mathfrak{m}\mathbf{I}_3 \end{pmatrix}$$

parallel axis theorem

and its time derivative

$$\dot{\mathbf{M}}_{\mathbb{I}} \triangleq \frac{d_{\mathbb{I}}\mathbf{M}_{\mathbb{I}}}{dt} = \overline{\mathcal{V}}_{\mathbb{I}}\mathbf{M}_{\mathbb{I}} - \mathbf{M}_{\mathbb{I}}\widetilde{\mathcal{V}}_{\mathbb{I}} = \overline{\mathcal{V}}_{\mathbb{I}}\mathbf{M}_{\mathbb{I}} + \mathbf{M}_{\mathbb{I}}\left(\overline{\mathcal{V}}_{\mathbb{I}}\right)^*$$

Equations of motion with $\beta_{\mathbb{I}} \triangleq {}^{\mathbb{I}}\mathcal{V}_{\mathbb{I}}$

The equations of motion with coordinates of the inertially referenced spatial velocity

$$\boxed{\mathfrak{f}_{\mathbb{I}} = \mathrm{M}_{\mathbb{I}}\dot{\beta}_{\mathbb{I}} + \mathfrak{b}_{\mathbb{I}}}$$

with gyroscopic term

$$\mathfrak{b}_{\mathbb{I}} \triangleq \dot{\mathrm{M}}_{\mathbb{I}}\mathcal{V}_{\mathbb{I}} = \overline{\mathcal{V}}_{\mathbb{I}}\mathrm{M}_{\mathbb{I}}\mathcal{V}_{\mathbb{I}}$$

Gyroscopic term with $\beta_{\mathbb{I}} \triangleq {}^{\mathbb{I}}\mathcal{V}_{\mathbb{I}}$

- Once again, the gyroscopic spatial force does <u>no work</u>
- Kinetic energy conservation easy to verify
- Inertially referenced spatial momentum is <u>conserved</u>.

Summary of equations of motion

The following summarizes the properties of the equations of motion from the different choice for generalized velocities

Formulation	$\mathbb{I}$ deriv, $\mathbb{C}$	$\mathbb{I}$ deriv, z	$\mathbb{B}$ deriv, z	Inertial ref, $\mathbb{I}$
Section	2.3.1	2.3.2	2.4	2.5
Gen. vel. β	${}^{\mathbb{I}}\mathcal{V}(\mathbb{C})$	${}^{\mathbb{I}}\mathcal{V}(z)$	${}^{\mathbb{B}}\mathcal{V}(z)$	$\mathcal{V}_{\mathbb{I}}$
Gyroscopic force $\mathfrak{b}$	$\overline{\mathcal{V}}^{\omega}(\mathbb{C})M(\mathbb{C})\mathcal{V}^{\omega}(\mathbb{C})$	$\overline{\mathcal{V}}^{\omega}(z)M(z)\mathcal{V}^{\omega}(z)$	$\overline{\mathcal{V}}(z)\mathfrak{h}(z)$	$\overline{\mathcal{V}}_{\mathbb{I}}\mathfrak{h}_{\mathbb{I}}$
Conserved spatial momentum	$\surd$			$\surd$
Non-working $\mathfrak{b}$	$\surd$		$\surd$	$\surd$
Independent of linear velocity	$\surd$	$\surd$		

Summary

- Looked at defining the dynamical system for multibody systems
- Looked at the choice of generalized coordinates, velocities and forces
- Developed equations of motion of a single rigid body using spatial notation
- Examined the impact of changing the generalized velocities on the equations of motion

SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics; composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
5. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
6. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
7. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity