Dynamics and Real-Time Simulation (DARTS) Laboratory

# Spatial Operator Algebra (SOA)

## *1. Spatial Notation*


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https://dartslab.jpl.nasa.gov/

Jet Propulsion Laboratory
California Institute of Technology

# SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics
5. **Mass matrix -** composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
6. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
7. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
8. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity

See https://dartslab.jpl.nasa.gov/References/index.php for publications and references on the SOA methodology.



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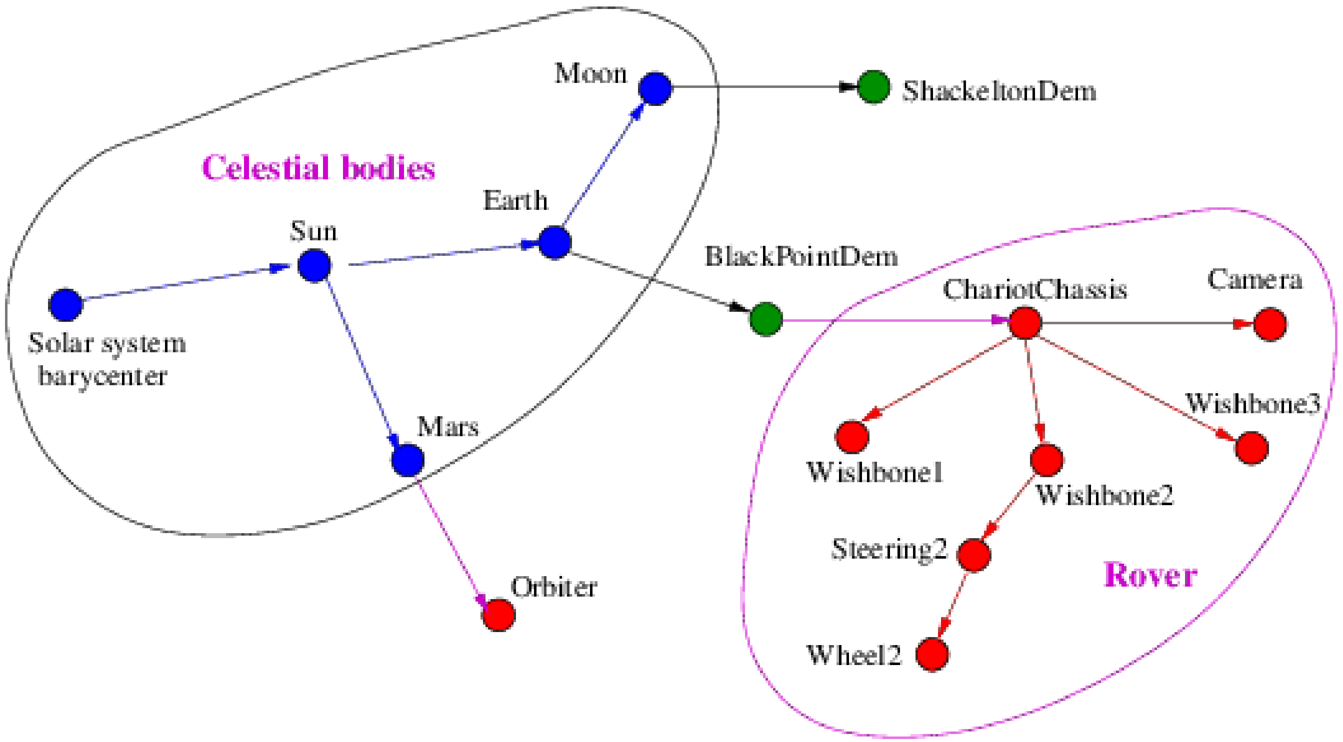
# Spatial Notation

# Outline

- Cover basics
- Introduce notational conventions
- Get comfortable working with multiple rotating frames
- Introduce 6D spatial notation

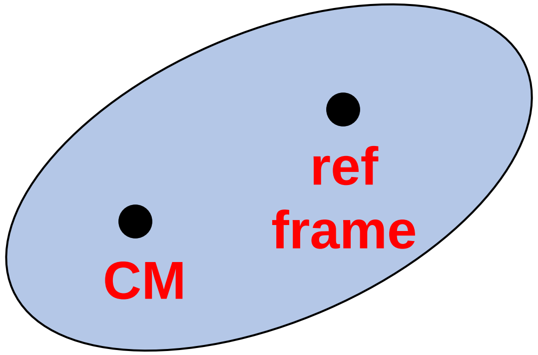
# Multibody Frames




- Examples of frames:
  - location of a thruster on the s/c bus
  - the motion of a pair of frames due to hinge articulation
  - the motion of the moon wrt to the earth; the earth wrt the sun etc.
- Frames have a *location* and *orientation*, i.e. a **pose**

# Linear/Angular properties

- When working with kinematics and dynamics, we often have to work with a combination of linear and angular properties
  - position/attitude
  - linear/angular velocities
  - force/moments
  - linear/angular momentum
  - mass/inertia
- Only at the body CM are the position & angular properties mostly decoupled
- However often have to work with general, non-CM reference frames – get messy coupling



# Spatial Velocities and Forces

- "**spatial**" notation – combines linear & angular terms together to help work with general (and not just CM) frames
    - position/attitude: homogeneous transform
    - velocities: spatial velocity [w, v] 6-dimensional
    - accelerations: like-wise
    - forces: spatial force [N, F] 6-dimensional

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

spatial velocity

$$\mathfrak{f} = \begin{bmatrix} N \\ F \end{bmatrix}$$

N → *moment*

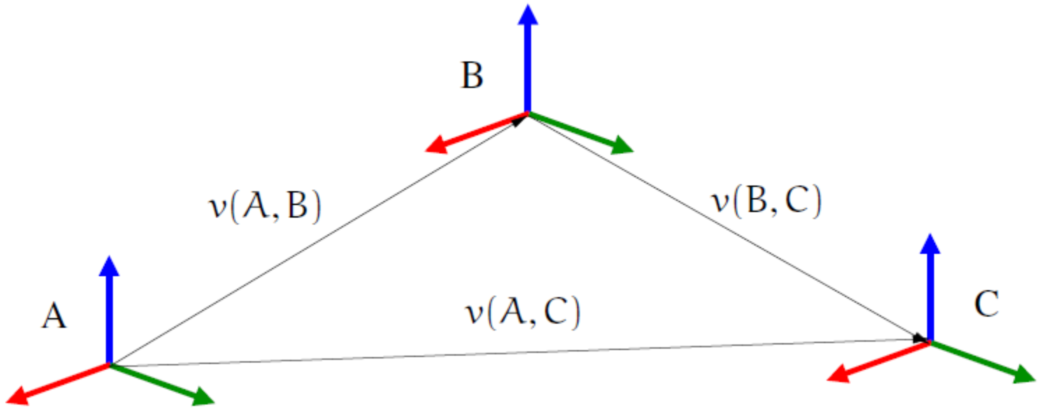
F → *force*

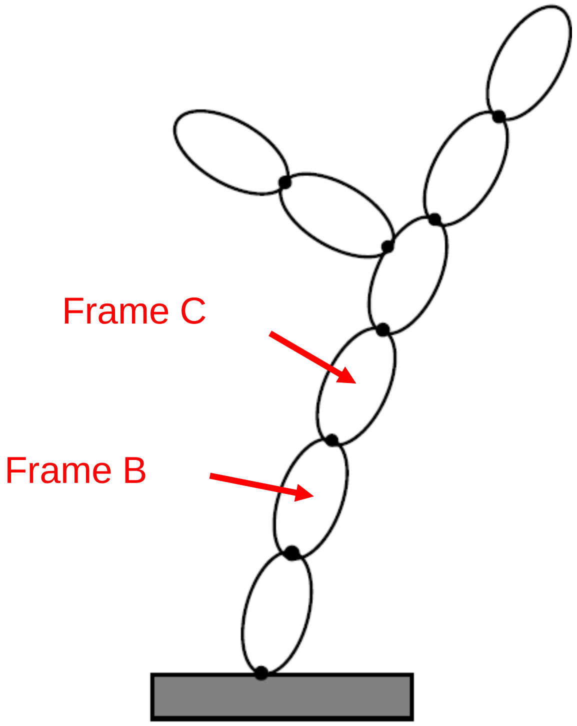
spatial force

*Not to be confused with twists/wrenches from classical kinematics theory, or with "spatial operators" coming up later*

# Focus on propagation relationships

Propagation relationships are building blocks for recursively computing body/node properties for articulated multibody systems



# Notational conventions

The * symbol is used to denote matrix transpose

$$A^* = A^T$$

The ~ symbol denotes the cross-product matrix

$$\tilde{l} = l^{\sim} \triangleq \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \quad \text{where} \quad l \triangleq \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

*Skew-symmetric* ***cross-product*** *matrix for a 3-vector*

$$l \otimes x = \tilde{l}\, x$$

# Spatial Transformations Recap

*Spatial notation offers concise & consistent transformation expressions for arbitrary non-CM points*

*rigid body transformation matrix*

Spatial velocities $${}^{C}\mathcal{V}(A,C) = \phi^*(B,C)\ {}^{B}\mathcal{V}(A,B)$$

Spatial forces $${}^{B}\mathfrak{f}(B) = \phi(B,C)\ {}^{C}\mathfrak{f}(C)$$

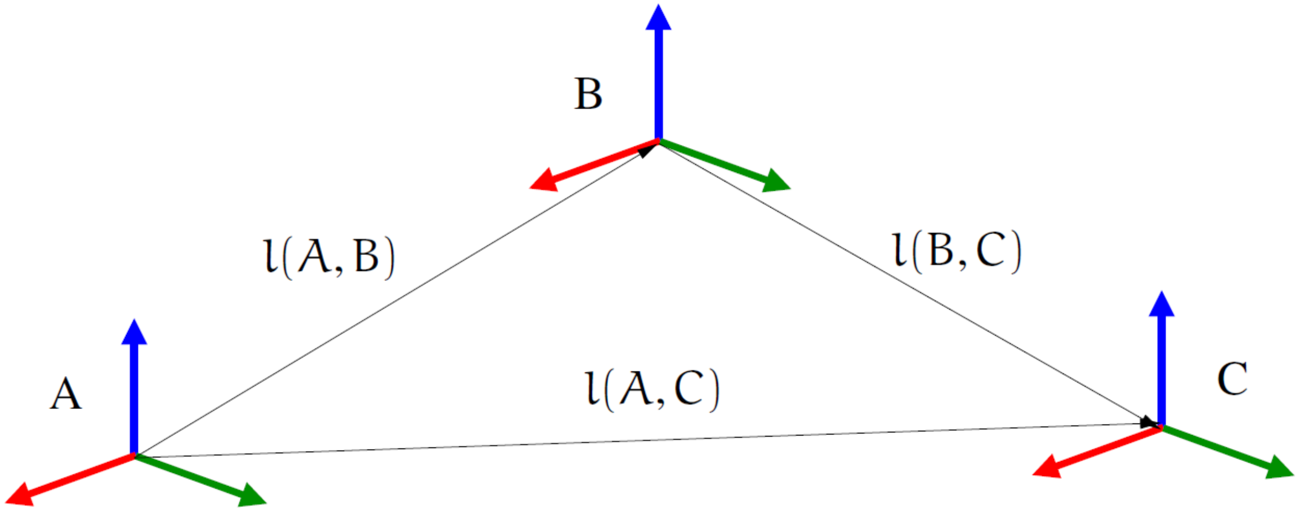
Spatial inertia $$M(x) = \phi(x,y)M(y)\phi^*(x,y)$$

Kinetic energy $$\mathfrak{K}_e = \frac{1}{2}\mathcal{V}^*(x)M(x)\mathcal{V}(x) = \frac{1}{2}\mathcal{V}^*(y)M(y)\mathcal{V}(y)$$

Spatial momentum $$\mathfrak{h}(x) = \phi(x,y)\mathfrak{h}(y)$$

# Position vectors

# Propagating Positions

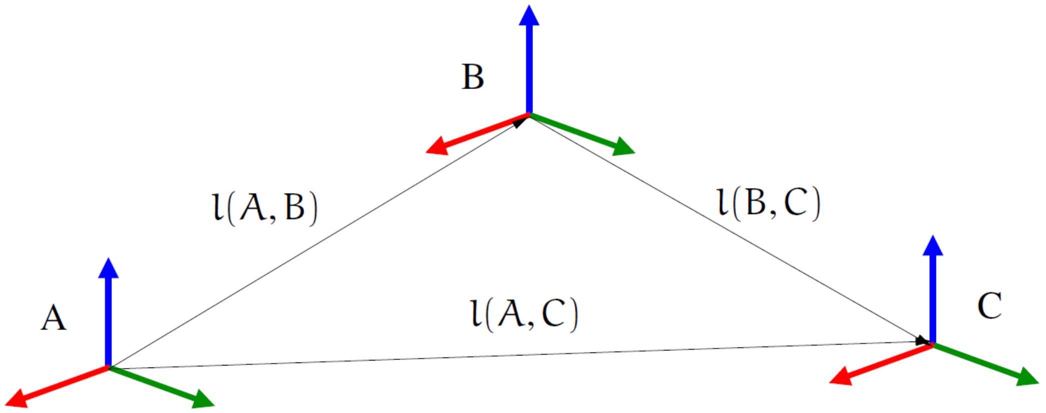



$$ {}^{A}\mathfrak{l}(A,C) = {}^{A}\mathfrak{l}(A,B) + \underbrace{{}^{A}\mathfrak{R}_{B}\;{}^{B}\mathfrak{l}(B,C)}_{{}^{A}\mathfrak{l}(B,C)} $$

*representation frame*

Can accumulate positional displacements – after representing in the same frame.

# Coordinate-free notation




$$^{A}\mathfrak{l}(A,C) = {}^{A}\mathfrak{l}(A,B) + {}^{A}\mathfrak{R}_{B}\, {}^{B}\mathfrak{l}(B,C)$$

$$\Updownarrow$$

$$\mathfrak{l}(A,C) = \mathfrak{l}(A,B) + \mathfrak{l}(B,C)$$

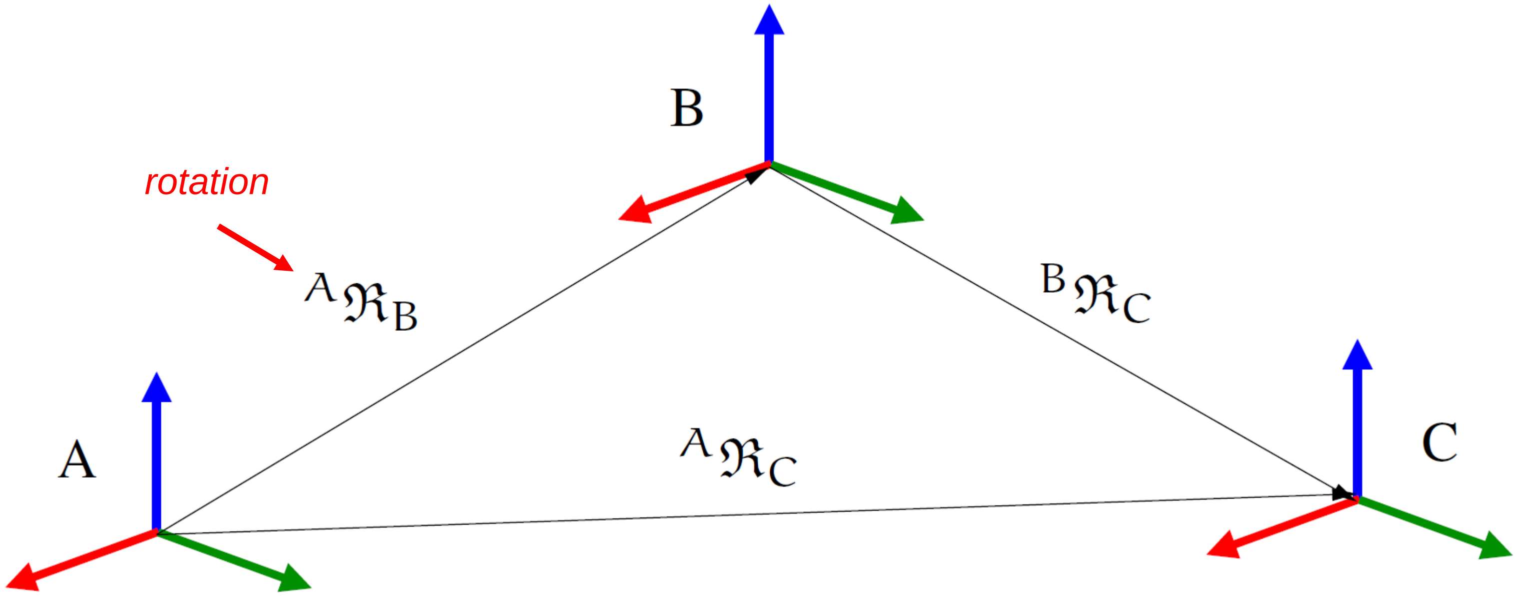
*coordinate-free notation*

Will use coordinate-free notation to reduce clutter from showing rotational transforms.

- Will show rotations when needed to avoid confusion

# Attitude Representations

# Accumulation of Rotations




$$^{A}\mathfrak{R}_{C} = {}^{A}\mathfrak{R}_{B}\, {}^{B}\mathfrak{R}_{C}$$

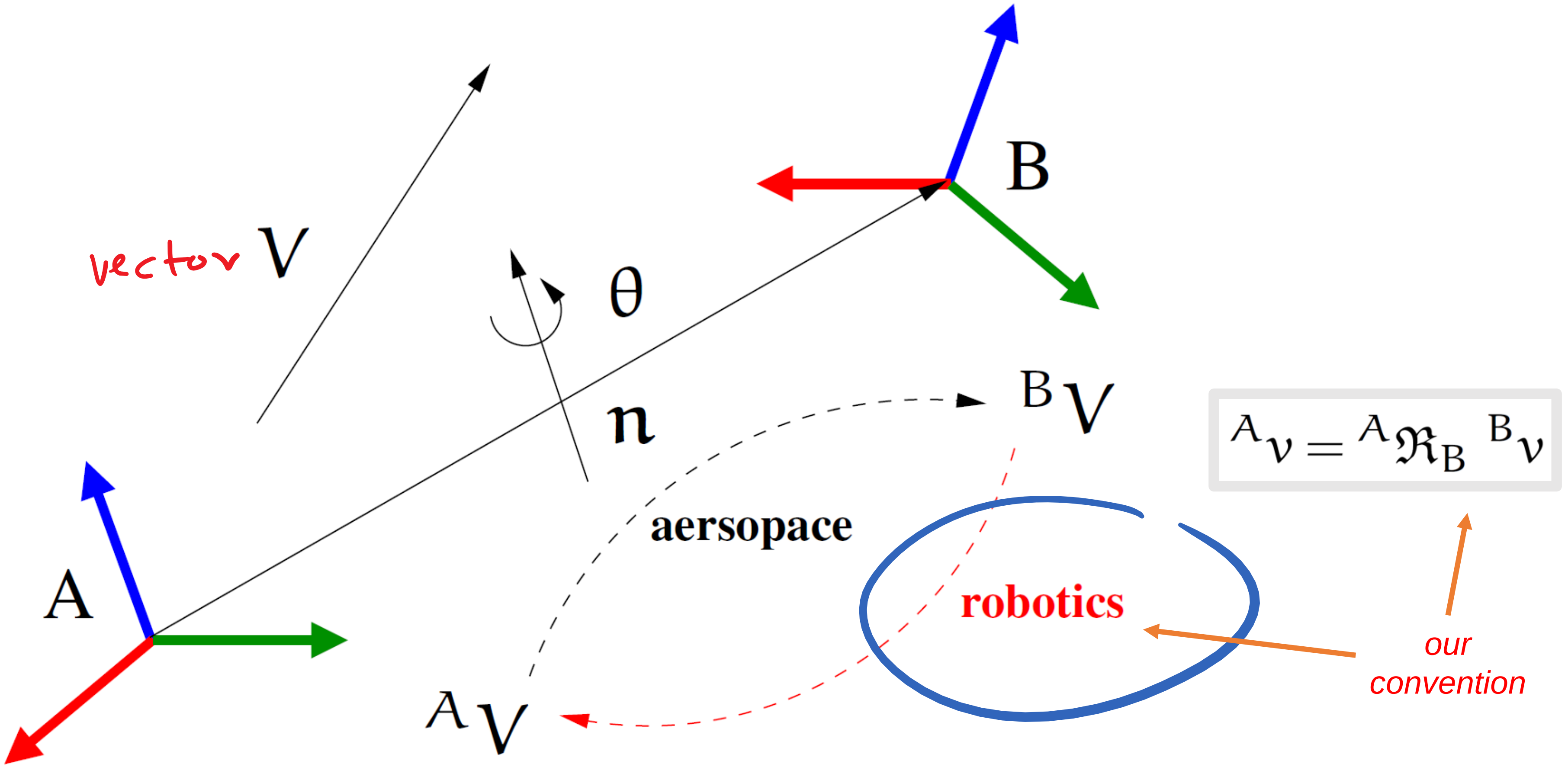
*indices must match when composing*

Can compute rotations by composing successive ones.

# Attitude representation conventions

- There are two parallel conventions for rotations, referred to as
  - **robotics** and **aerospace** rotation conventions
  - **passive** and **active** rotation conventions
  - **right** and **left** multiplication convention
- Both are valid in their own right, but almost exactly opposite of each other
- Need to be careful when working with both simultaneously to avoid misinterpretation and incorrect use

# Aerospace vs Robotics Convention

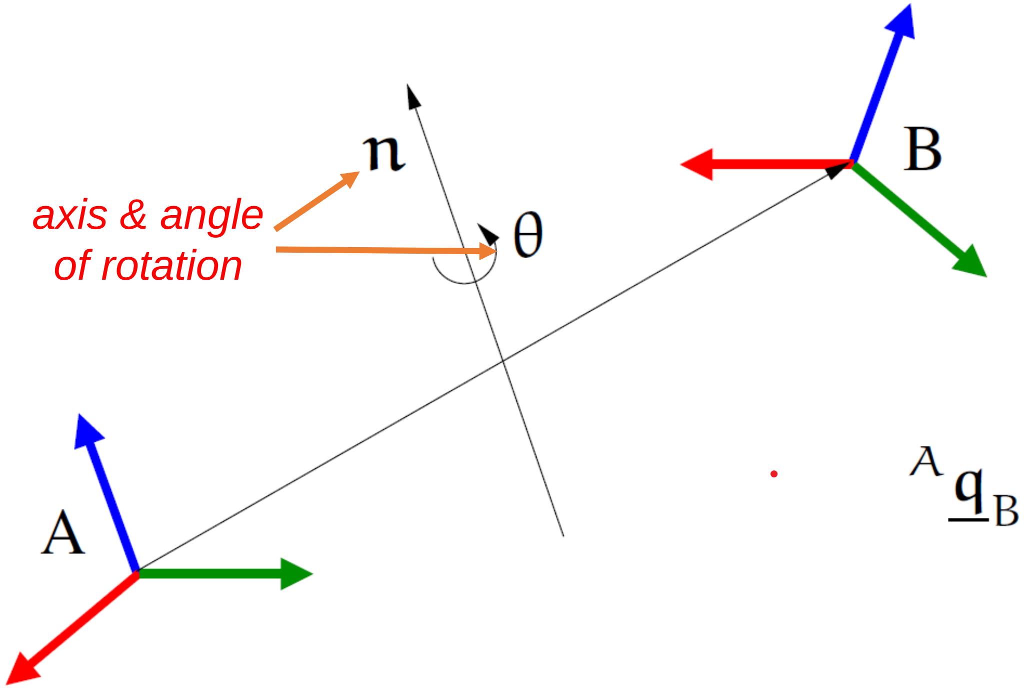


# Attitude representations

- **General result:** minimal (3 scalar) attitude representations *cannot both be global and non-singular*.
  - Euler angle & Rodrigues 3-parameters are global, but singular
  - Cayley 3-parameter representations are non-singular, but not global
- Unit quaternion 4-parameter representations mostly avoid trigonometric terms and are a good choice for transformations

# Some Attitude Representations

The attitude of frame B with respect to frame A can be defined as a rotation about a fixed axis




$$ {}^{A}\underline{\mathbf{q}}_{B} = \begin{bmatrix} q \\ q_0 \end{bmatrix} = \begin{bmatrix} \sin(\theta/2)\mathbf{n} \\ \cos(\theta/2) \end{bmatrix} $$

(unit quaternion)

*Rodrigues parameters*

$$ {}^{A}\mathfrak{R}_{B} = \exp(\tilde{\mathbf{n}}\theta) $$ (exponential formula)

$$ = \cos(\theta) I_3 + [1 - \cos(\theta)]\mathbf{n}\mathbf{n}^* + \sin(\theta)\,\tilde{\mathbf{n}} $$ (angle/axis)

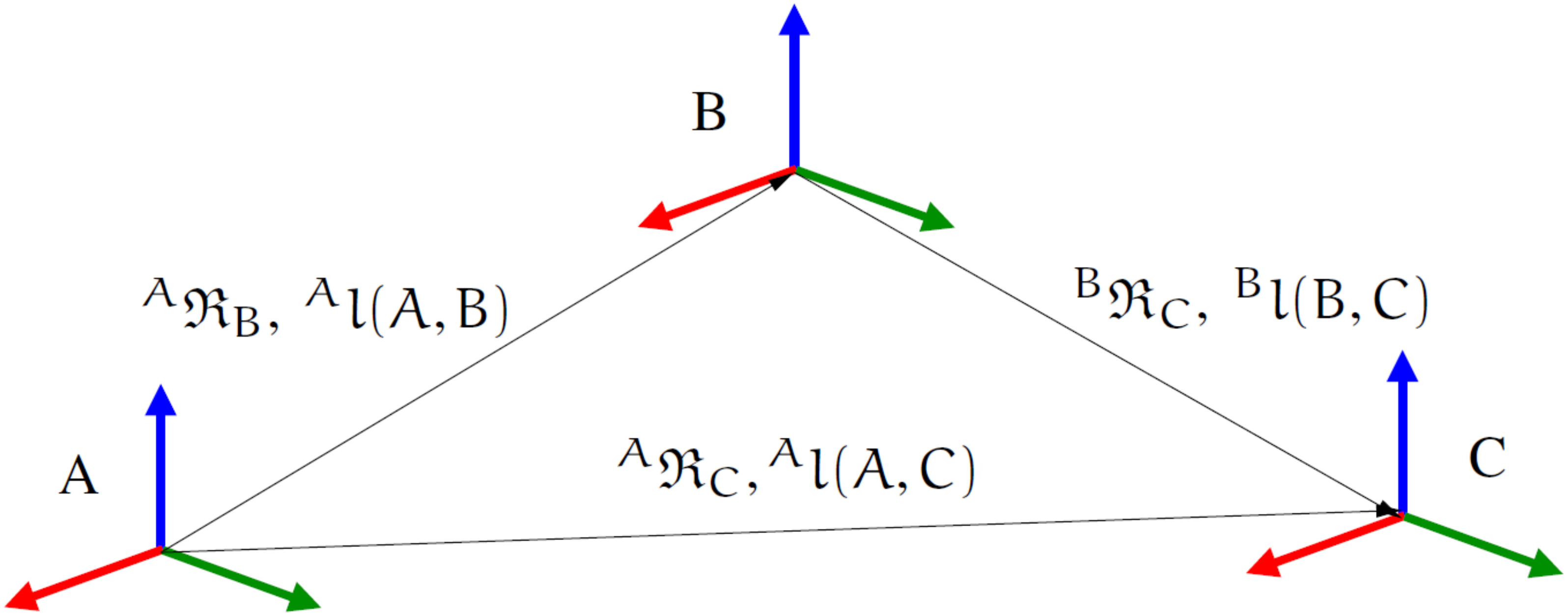
# Multi-purpose attitude representations

- **Unit Quaternions** $${}^{A}\underline{q}_{B} = \begin{bmatrix} q \\ q_0 \end{bmatrix} = \begin{bmatrix} \sin(\theta/2)\mathbf{n} \\ \cos(\theta/2) \end{bmatrix}$$
  - Great for applying *rotational transformations* across frames
- **Rodrigues parameters** $$u \triangleq \mathbf{n}\theta$$
  - Good 3-parameter representation for *integrating attitude rates*
  - though have to re-center coordinates periodically to avoid singularity

# Homogeneous Transformations

# Working with both position/rotations

Typically have to deal with both rotations and displacements (i.e. **poses**) simultaneously.



# Homogeneous Transform

Combined attitude and position information is also referred to as "**pose**". A pose can be represented as a 4x4 **homogeneous transform** matrix

$$ {}^{A}\mathbb{T}_{B} = \begin{pmatrix} {}^{A}\mathfrak{R}_{B} & {}^{A}\mathfrak{l}(A,B) \\ 0 & 1 \end{pmatrix} $$

homogeneous transform

**Inverse:** $$ {}^{F}\mathrm{T}_{G}^{-1} = {}^{G}\mathrm{T}_{F} $$

# ${}^{F}\mathrm{T}_{G}$ inverse

**Claim:** $$\boxed{{}^{F}\mathrm{T}_{G}^{-1} = {}^{G}\mathrm{T}_{F}}$$

**Verification:**

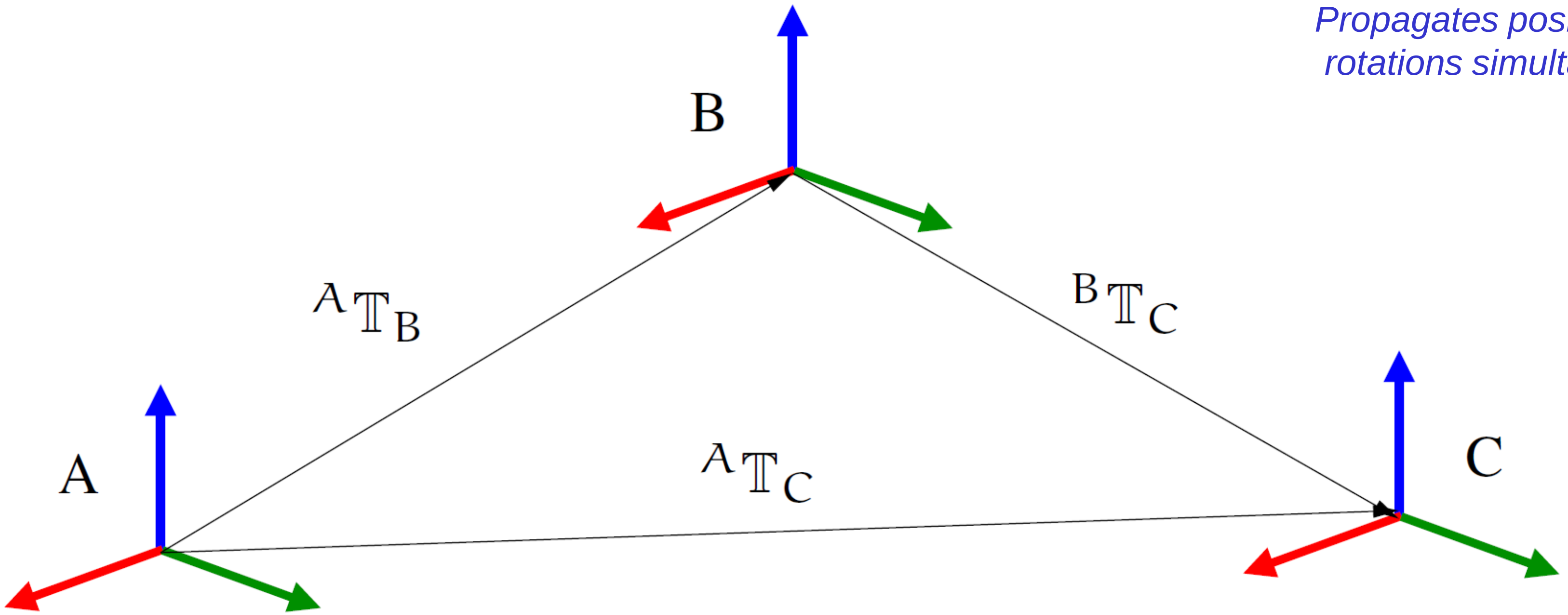
$$
{}^{G}\mathrm{T}_{F}\,{}^{F}\mathrm{T}_{G} = \overbrace{\begin{pmatrix} {}^{G}\mathfrak{R}_{F} & {}^{G}\mathfrak{l}(G,F) \\ 0 & 1 \end{pmatrix}}^{{}^{G}T_{F}} \overbrace{\begin{pmatrix} {}^{F}\mathfrak{R}_{G} & {}^{F}\mathfrak{l}(F,G) \\ 0 & 1 \end{pmatrix}}^{{}^{F}T_{G}}
$$

$$
= \begin{pmatrix} \mathbf{I}_3 & {}^{G}\mathfrak{l}(F,G) + {}^{G}\mathfrak{l}(G,F) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_3 & 0 \\ 0 & 1 \end{pmatrix}
$$

# Propagating pose across frames

Computing overall relative pose by composing component poses (like rotations)

*Propagates positions and rotations simultaneously*




$${}^{A}\mathbb{T}_{B} \qquad {}^{B}\mathbb{T}_{C} \qquad {}^{A}\mathbb{T}_{C}$$

$${}^{A}\mathbb{T}_{C} = {}^{A}\mathbb{T}_{B}\,{}^{B}\mathbb{T}_{C}$$

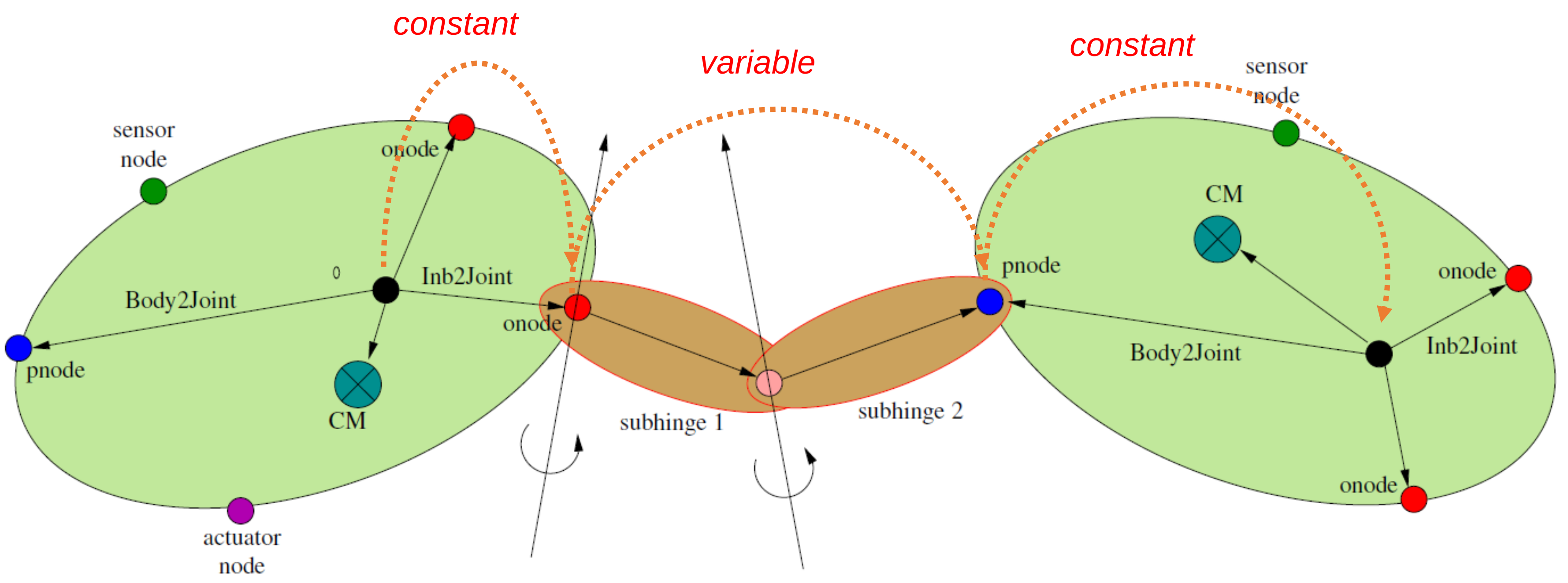
*Composing homog. transforms*

$$\begin{bmatrix} {}^{A}l(A,C) \\ 1 \end{bmatrix} = {}^{A}\mathbb{T}_{B} \begin{bmatrix} {}^{B}l(B,C) \\ 1 \end{bmatrix}$$

*relative position*

# Propagating Poses

- Frequent need to compute poses of bodies and frames with respect to each other and the inertial frame.



# Composing Homog. Transforms

**Claim:** $${}^{\mathbb{A}}\mathbb{T}_{\mathbb{C}} = {}^{\mathbb{A}}\mathbb{T}_{\mathbb{B}}\,{}^{\mathbb{B}}\mathbb{T}_{\mathbb{C}}$$

**Verification:**

$${}^{\mathbb{F}}\mathbb{T}_{\mathbb{G}}\,{}^{\mathbb{G}}\mathbb{T}_{\mathbb{H}} = \begin{pmatrix} {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}} & {}^{\mathbb{F}}\mathfrak{l}(\mathbb{F},\mathbb{G}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{\mathbb{G}}\mathfrak{R}_{\mathbb{H}} & {}^{\mathbb{G}}\mathfrak{l}(\mathbb{G},\mathbb{H}) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{H}} & {}^{\mathbb{F}}\mathfrak{l}(\mathbb{F},\mathbb{G}) + {}^{\mathbb{F}}\mathfrak{l}(\mathbb{G},\mathbb{H}) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{H}} & {}^{\mathbb{F}}\mathfrak{l}(\mathbb{F},\mathbb{H}) \\ 0 & 1 \end{pmatrix} = {}^{\mathbb{F}}\mathbb{T}_{\mathbb{H}}$$

# Computing relative position

Claim:

$$\begin{bmatrix} {}^{\mathbb{A}}\mathfrak{l}(\mathbb{A}, \mathbb{C}) \\ 1 \end{bmatrix} = {}^{\mathbb{A}}\mathbb{T}_{\mathbb{B}} \begin{bmatrix} {}^{\mathbb{B}}\mathfrak{l}(\mathbb{B}, \mathbb{C}) \\ 1 \end{bmatrix}$$

Verification:

Follows from

$${}^{\mathbb{F}}\mathfrak{p}(\mathbb{F}, \mathbb{O}) = {}^{\mathbb{F}}\mathfrak{l}(\mathbb{F}, \mathbb{G}) + {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}}\, {}^{\mathbb{G}}\mathfrak{p}(\mathbb{G}, \mathbb{O})$$

$${}^{\mathbb{A}}\mathbb{T}_{\mathbb{B}} = \begin{pmatrix} {}^{\mathbb{A}}\mathfrak{R}_{\mathbb{B}} & {}^{\mathbb{A}}\mathfrak{l}(\mathbb{A}, \mathbb{B}) \\ 0 & 1 \end{pmatrix}$$

# Time Derivatives of Vectors

# Time derivative of a 3-vector

- We can only differentiate coordinate representations of vector quantities
- So we need to specify which frame the coordinates are represented in , i.e. which frame we are differentiating or observing in, eg.

$$\frac{{}^{\mathbb{F}}\mathrm{d}_{\mathbb{F}}\, x(s)}{\mathrm{d}s} \triangleq \frac{\mathrm{d}\left[{}^{\mathbb{F}}x(s)\right]}{\mathrm{d}s} = \begin{bmatrix} \frac{\mathrm{d}x_1(s)}{\mathrm{d}s} \\ \frac{\mathrm{d}x_2(s)}{\mathrm{d}s} \\ \frac{\mathrm{d}x_3(s)}{\mathrm{d}s} \end{bmatrix}$$

- *Resulting derivative is itself a 3-vector*

# Time derivative representations

- The derivative in different frames are not necessarily the same

$$\frac{\mathrm{d}_{\mathbb{F}}\mathrm{x}}{\mathrm{d}s} \neq \frac{\mathrm{d}_{\mathbb{G}}\mathrm{x}}{\mathrm{d}s}$$

- The time derivative of a vector is itself a vector
  - and thus it can be represented in a frame other than the derivative frame

representation frame

derivative frame

$$\frac{{}^{\mathbb{G}}\mathrm{d}_{\mathbb{F}}\mathrm{x}}{\mathrm{d}s} = {}^{\mathbb{G}}\mathfrak{R}_{\mathbb{F}} \frac{{}^{\mathbb{F}}\mathrm{d}_{\mathbb{F}}\mathrm{x}}{\mathrm{d}s}$$

# Angular velocity

- Angular velocity is defined via the time derivative property of rotations:

$$\frac{\mathrm{d}\,{}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}}}{\mathrm{d}t} = {}^{\mathbb{F}}\tilde{\omega}(\mathbb{F},\mathbb{G})\;{}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}} = {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}}\;{}^{\mathbb{G}}\tilde{\omega}(\mathbb{F},\mathbb{G})$$

angular velocity

# Relating different time derivatives

*If have time derivative of a vector in one frame, can we get its time derivative in a different frame?*

$$\frac{\mathrm{d}_{\mathbb{F}}\mathrm{x}}{\mathrm{dt}} = \frac{\mathrm{d}\ {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}}\ {}^{\mathbb{G}}\mathrm{x}}{\mathrm{dt}} = {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}} \frac{\mathrm{d}_{\mathbb{G}}\mathrm{x}}{\mathrm{dt}} + {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}}\ {}^{\mathbb{G}}\tilde{\omega}\,(\mathbb{F},\mathbb{G})\ {}^{\mathbb{G}}\mathrm{x}$$

*or simply*

$$\boxed{\frac{\mathrm{d}_{\mathbb{F}}\mathrm{x}}{\mathrm{dt}} = \frac{\mathrm{d}_{\mathbb{G}}\mathrm{x}}{\mathrm{dt}} + \tilde{\omega}(\mathbb{F},\mathbb{G})\mathrm{x}}$$

*The derivatives in different frames are the same only if there is no relative angular velocity between the frames.*

# Linear Velocities

# Linear velocities

- Linear velocity is the time derivative of position vector in the initial frame:

$$v(x,y) = \frac{d_x \mathfrak{l}(x,y)}{dt}$$

- Linear velocity reversal *(show)*

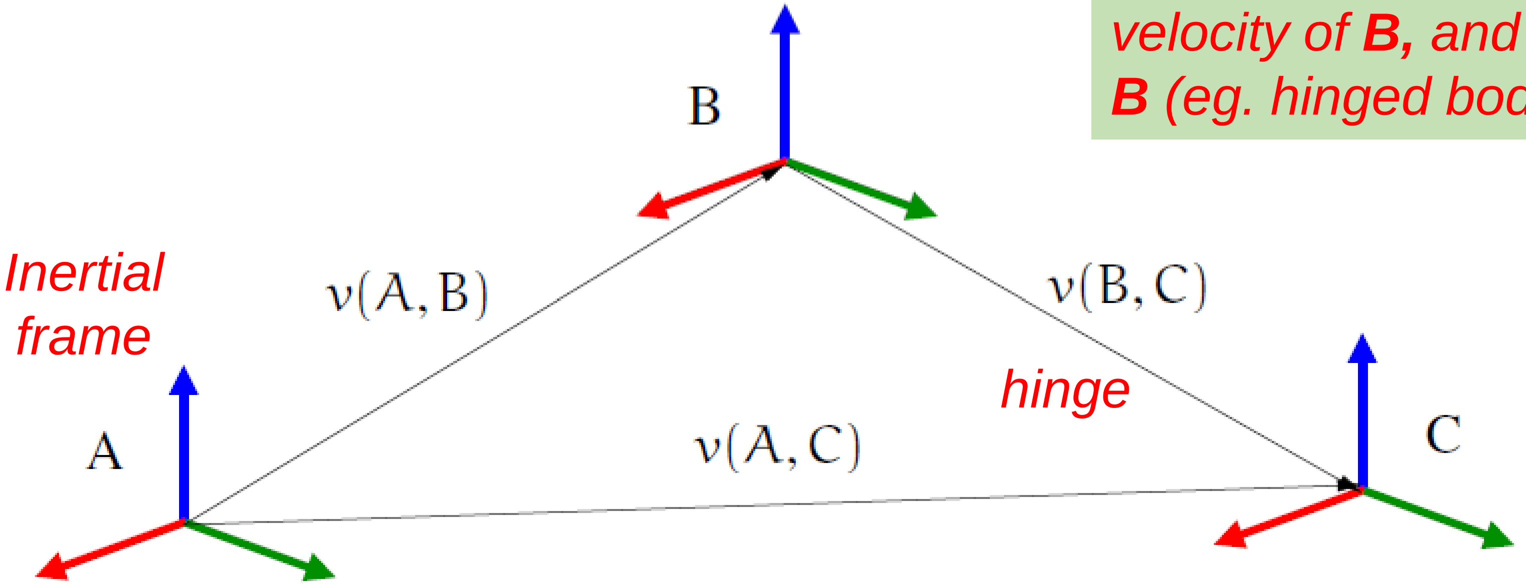
$$v(y,x) = -v(x,y) + \underline{\tilde{\omega}(x,y)\, v(x,y)}$$

*vanishes when relative angular velocity is zero*

# Accumulating linear velocities

*Can accumulate linear velocities across multiple frames*

***Example:* *A*** *is inertial frame, and want inertial linear velocity of* ***C*** *given velocity of* ***B,*** *and* ***C*** *wrt* ***B*** *(eg. hinged bodies)*

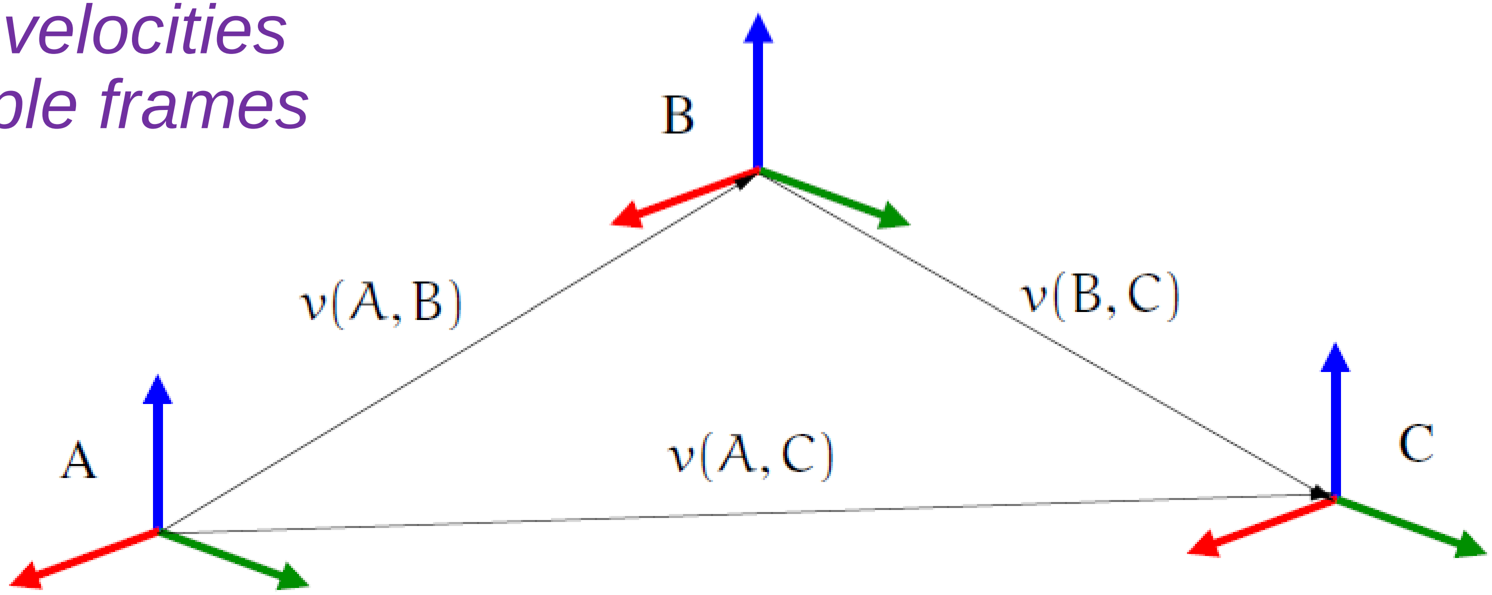



$$v(A,C) = v(A,B) + \tilde{\omega}(A,B)\ l(B,C) + v(B,C)$$

# Spatial Velocity

# Accumulating linear/angular velocities

*Accumulating both linear and angular velocities across multiple frames*

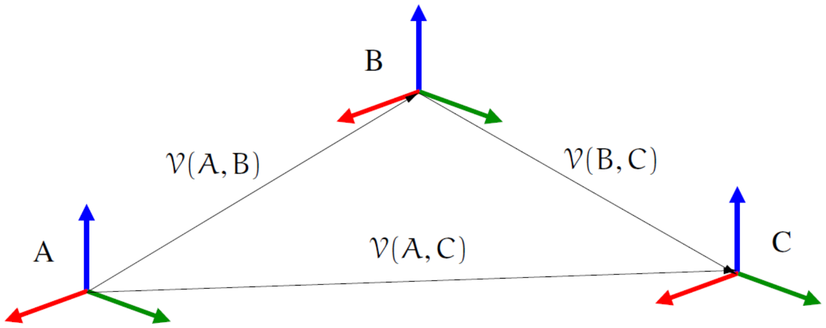



$$\omega(A, C) = \omega(A, B) + \omega(B, C)$$

$$v(A, C) = v(A, B) + \tilde{\omega}(A, B)\, l(B, C) + v(B, C)$$

# Accumulating Spatial velocities

*Can accumulate spatial velocities across multiple frames*




$$\omega(A,C) = \omega(A,B) + \omega(B,C)$$

$$v(A,C) = v(A,B) + \tilde{\omega}(A,B)\, l(B,C) + v(B,C)$$

spatial velocity propagation

$$\mathcal{V}(A,C) = \phi^*(B,C)\, \mathcal{V}(A,B) + \mathcal{V}(B,C)$$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

spatial velocity (6D)

$$\phi(B,C) = \begin{pmatrix} I_3 & \tilde{l}(B,C) \\ 0 & I_3 \end{pmatrix}$$

rigid body transformation matrix (6x6)

Uses components of the **T(B,C)** homogeneous transform

# Spatial velocity propagation

Claim:

$$\mathcal{V}(A,C) = \phi^*(B,C)\,\mathcal{V}(A,B) + \mathcal{V}(B,C)$$

Verification:

$$\omega(A,C) = \omega(A,B) + \omega(B,C)$$

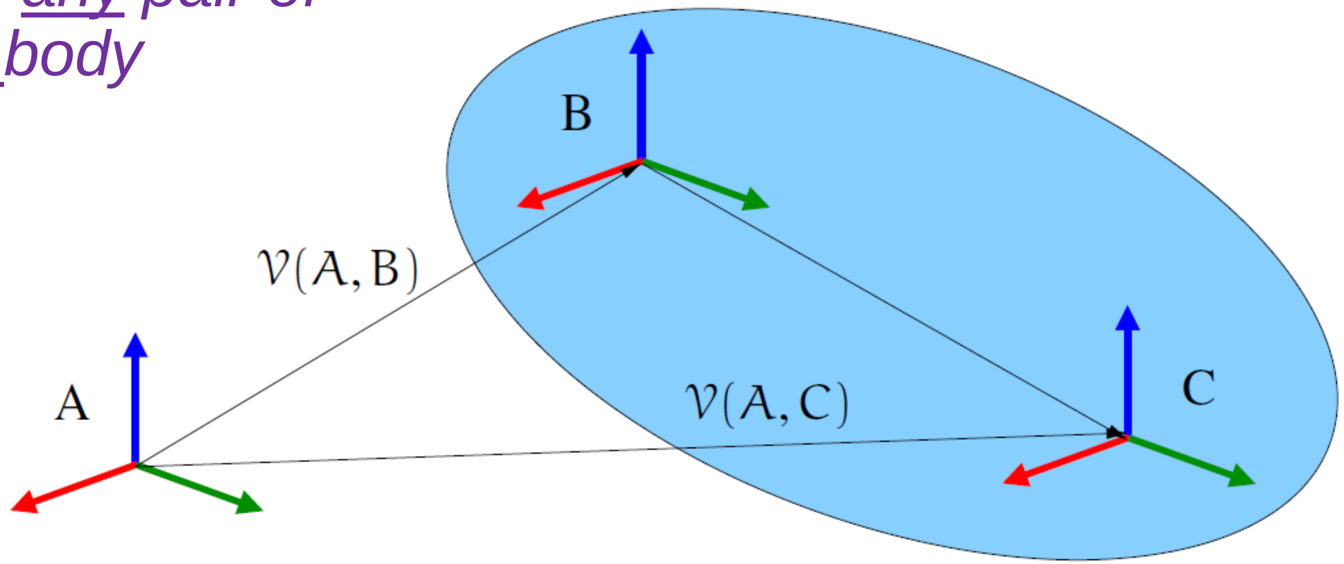
$$\begin{aligned} \nu(A,C) &= \nu(A,B) + \tilde{\omega}(A,B)\,l(B,C) + \nu(B,C) \\ &= -\tilde{l}(B,C)\,\omega(A,B) + \nu(A,B) + \nu(B,C) \end{aligned}$$

$$\mathcal{V} = \begin{bmatrix} \omega \\ \nu \end{bmatrix}$$

$$\phi^*(x,y) = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ -\tilde{l}(x,y) & \mathbf{I}_3 \end{pmatrix}$$

# Special case: Transforming spatial velocities across a rigid body

*Simple way to transform spatial velocities across any pair of points on a rigid body*



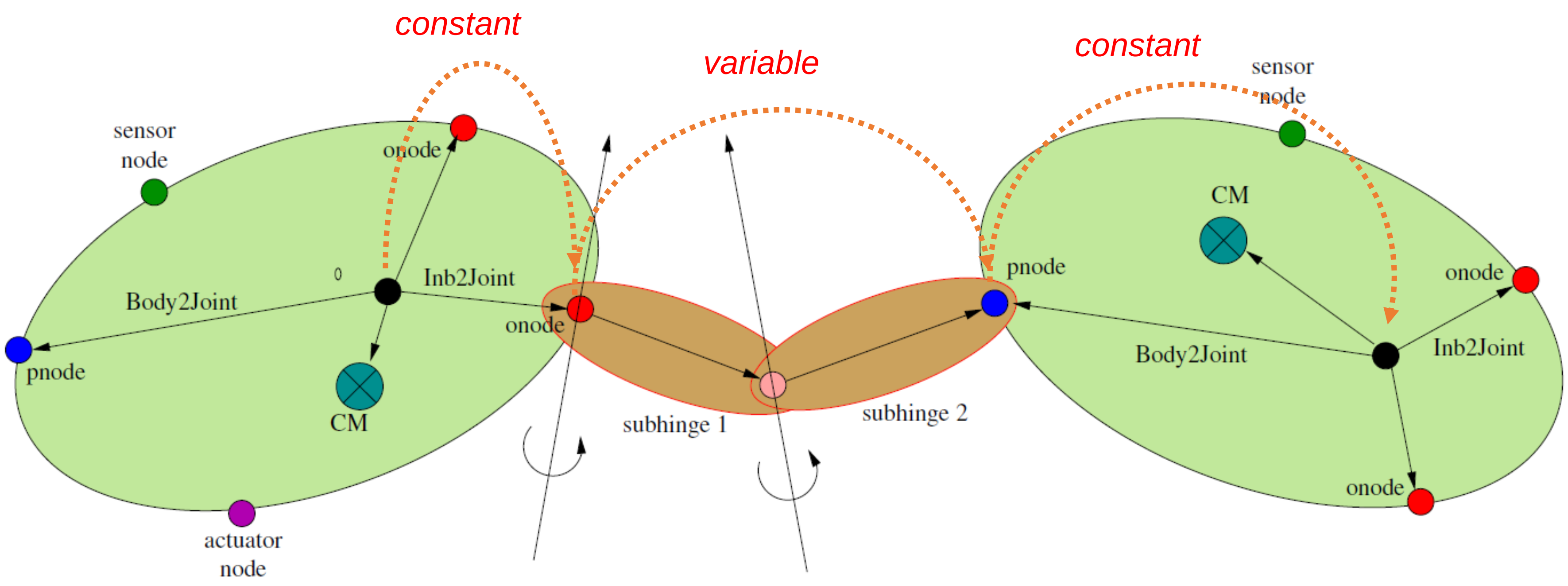

$$\mathcal{V}(A,C) = \phi^*(B,C)\,\mathcal{V}(A,B) + \mathcal{V}(B,C)$$

must match

$$\mathcal{V}(A,C) = \phi^*(B,C)\,\mathcal{V}(A,B)$$

# Propagating Spatial Velocities

We can use the spatial velocity propagation relationships for computing spatial velocities of bodies and frames.



# Structure of $\phi(\mathbb{F},\mathbb{G})$

- We have been using the coordinate free representations so far.

$$\phi(\mathbb{F},\mathbb{G}) = \begin{pmatrix} \mathbf{I}_3 & \tilde{\mathfrak{l}}(\mathbb{F},\mathbb{G}) \\ \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix}$$

skew-symmetric

zero

identity

- The full, explicit form which includes rotations is:

$$\phi(\mathbb{F},\mathbb{G}) = \begin{pmatrix} \mathbf{I}_3 & {}^{\mathbb{F}}\tilde{\mathfrak{l}}(\mathbb{F},\mathbb{G}) \\ \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix} \begin{pmatrix} {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}} & \mathbf{0}_3 \\ \mathbf{0}_3 & {}^{\mathbb{F}}\mathfrak{R}_{\mathbb{G}} \end{pmatrix}$$

# Properties of $\phi(x, y)$

*Properties are very similar to those for rotational matrices.*

$$\phi(x, x) = \mathbf{I}_6$$ *Identity*

must match

$$\phi(x, z) = \phi(x, y)\phi(y, z)$$ *Products*

$$\begin{pmatrix} I & \tilde{\ell}(x, z) \\ 0 & I \end{pmatrix}$$

$$\phi^{-1}(x, y) = \phi(y, x)$$ *Inverse*

$$\begin{pmatrix} I & \tilde{\ell}(y, x) \\ 0 & I \end{pmatrix}$$

# Product rule

Claim: $\phi(x, z) = \phi(x, y)\phi(y, z)$

Verification:

$$\phi(x,y)\phi(y,z) = \underbrace{\begin{pmatrix} \mathbf{I}_3 & \tilde{l}(x,y) \\ \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix}}_{\phi(x,y)} \underbrace{\begin{pmatrix} \mathbf{I}_3 & \tilde{l}(y,z) \\ \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix}}_{\phi(y,z)}$$

$$= \begin{pmatrix} \mathbf{I}_3 & \tilde{l}(x,y) + \tilde{l}(y,z) \\ \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_3 & \tilde{l}(x,z) \\ \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix} = \phi(x,z)$$

# Reversing spatial velocity

- Spatial velocity reversal *(show)*

$$\mathcal{V}(y,x) = -\phi^*(y,x)\mathcal{V}(x,y)$$

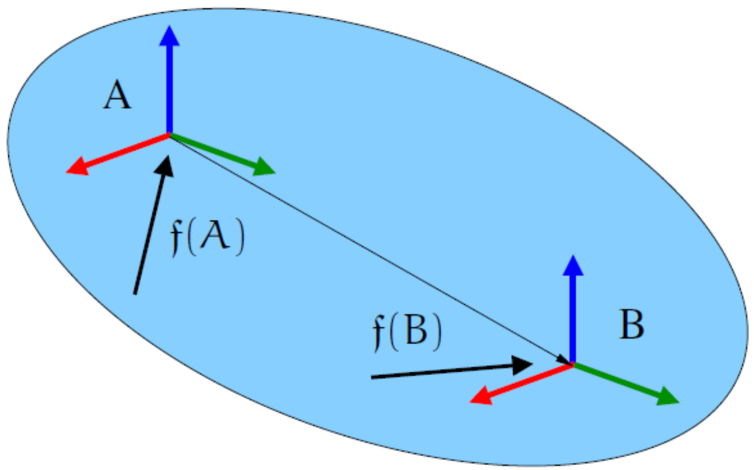
- *This is a generalization of linear velocity reversal seen earlier*

$$v(y,x) = -v(x,y) + \tilde{\omega}(x,y)\, v(x,y)$$

# Spatial Forces

# Force and moments

Can transform **forces** and **moments** across points on a rigid body




*moments*

$$N(A) = N(B) + \tilde{l}(A,B)\, F(B)$$

$$F(A) = F(B)$$

*forces*

$$\mathfrak{f} = \begin{bmatrix} N \\ F \end{bmatrix}$$

spatial force

$$\mathfrak{f}(A) = \phi(A,B)\, \mathfrak{f}(B)$$

# Show force transformation

Claim:

$$\mathfrak{f}(\mathcal{A}) = \phi(\mathcal{A},\mathcal{B})\ \mathfrak{f}(\mathcal{B})$$

Verification:

$$\mathfrak{f} = \begin{bmatrix} N \\ F \end{bmatrix}$$

$$\begin{aligned} N(\mathcal{A}) &= N(\mathcal{B}) + \tilde{l}(\mathcal{A},\mathcal{B})\ F(\mathcal{B}) \\ F(\mathcal{A}) &= F(\mathcal{B}) \end{aligned}$$

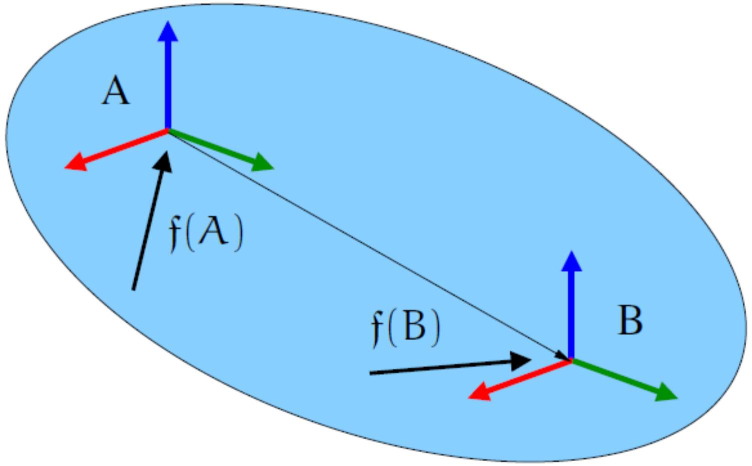
$$\phi(\mathcal{A},\mathcal{B}) = \begin{pmatrix} I_3 & \tilde{l}(\mathcal{A},\mathcal{B}) \\ 0 & I_3 \end{pmatrix}$$

# Rigid body dual relationships

Dual transformations for
spatial velocities and forces

$$\mathcal{V}(A) = \phi^*(B, A)\ \mathcal{V}(B)$$

$$f(A) = \phi(A, B)\ f(B)$$




$$\mathcal{V}(x) = \mathcal{V}(I, x)$$

*Omit inertial frame for notational short hand when redundant*

$$f^*(A)\ \mathcal{V}(A) = f^*(B)\ \mathcal{V}(B)$$

Power relationship is invariant to location

# Accumulating Spatial Forces

- There is often a need to compute the overall spatial force on a rigid body, eg. from attached actuators
- The process is to transform each of the forces to a common point, and then sum them up as follows

$$ {}^{B}\mathfrak{f} = \sum_{i} \underbrace{\phi(B,i)\,\mathfrak{f}(i)} $$

*overall spatial force at B*

*transform spatial force to B*

# Spatial Cross-Product

# Spatial cross product

- Cross product for spatial vectors (6-vectors)

$$z \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \qquad \tilde{z} \triangleq \begin{pmatrix} \tilde{x} & 0_3 \\ \tilde{y} & \tilde{x} \end{pmatrix}$$

$$z \otimes c \triangleq \tilde{z}c = \begin{pmatrix} \tilde{x}a \\ \tilde{y}a + \tilde{x}b \end{pmatrix} \qquad c \triangleq \begin{bmatrix} a \\ b \end{bmatrix}$$

*spatial vector cross-product*

$\tilde{z}z = 0$

# Spatial cross product identities

- Cross product identities similar to 3D case

$$\widetilde{\mathrm{A}}\mathrm{A} = 0$$

$$\widetilde{\mathrm{A}}\mathrm{B} = -\widetilde{\mathrm{B}}\mathrm{A} \qquad \textit{(skew-symmetry)}$$

$$\widetilde{\mathrm{A}}\,\widetilde{\mathrm{B}}\mathrm{C} + \widetilde{\mathrm{B}}\,\widetilde{\mathrm{C}}\mathrm{A} + \widetilde{\mathrm{C}}\,\widetilde{\mathrm{A}}\mathrm{B} = 0 \qquad \textit{(Jacobi identity)}$$

$$\widetilde{\mathrm{A}}\,\widetilde{\mathrm{B}} - \widetilde{\mathrm{B}}\,\widetilde{\mathrm{A}} = \widetilde{\widetilde{\mathrm{A}}\mathrm{B}} \qquad \textit{(commutator)}$$

# Spatial cross product related $\bar{\mathfrak{z}}$ matrix

- Unlike 3D cross products

$$-\tilde{\mathfrak{z}}^* \neq \tilde{\mathfrak{z}}$$

- Define

$$\bar{\mathfrak{z}} \triangleq \begin{pmatrix} \tilde{\mathfrak{x}} & \tilde{\mathfrak{y}} \\ \mathbf{0}_3 & \tilde{\mathfrak{x}} \end{pmatrix} = -\tilde{\mathfrak{z}}^*$$

# Points on a rigid body

For a pair of points x, y fixed to a **rigid** body

$$\mathcal{V}(y) = \phi^*(x,y)\, \mathcal{V}(x)$$

For this, the following identities are true:

$$\phi^*(x,y)\, \widetilde{\mathcal{V}}(x) = \widetilde{\mathcal{V}}(y) \phi^*(x,y)$$
$$\overline{\mathcal{V}(x)}\, \phi(x,y) = \phi(x,y)\, \overline{\mathcal{V}(y)}$$

# $\phi(x, y)$ & spatial cross-products

- Some identities

$$[\phi^*(x, y)\ X]^\sim = \phi^*(x, y)\, \widetilde{X}\, \phi^{-*}(x, y)$$

$$[\phi^*(x, y)\ X]^\sim\, \phi^*(x, y) = \phi^*(x, y)\, \widetilde{X}$$

# Spatial Accelerations

# Spatial accelerations

- The **spatial acceleration** is the time derivative of a spatial velocity with respect to a frame H defined as

$$\alpha_H(F, G) \triangleq \frac{d_H V(F, G)}{dt}$$

- Common choices for the **H** frame are
  - the inertial frame I
  - the "from" frame F
  - the "to" frame G

# Coriolis term expressions

- H = A $$\mathfrak{a} = \begin{bmatrix} 0 \\ \tilde{\omega}(x)\left[v(y) - v(x)\right] \end{bmatrix}$$
- H = B $$\mathfrak{a} = \begin{bmatrix} \tilde{\omega}(x)\omega(x,y) \\ \tilde{\omega}(x)\left[v(y) - v(x) + v(x,y)\right] \end{bmatrix}$$
- H = C $$\mathfrak{a} = \tilde{\mathcal{V}}(y)\mathcal{V}(x,y)$$

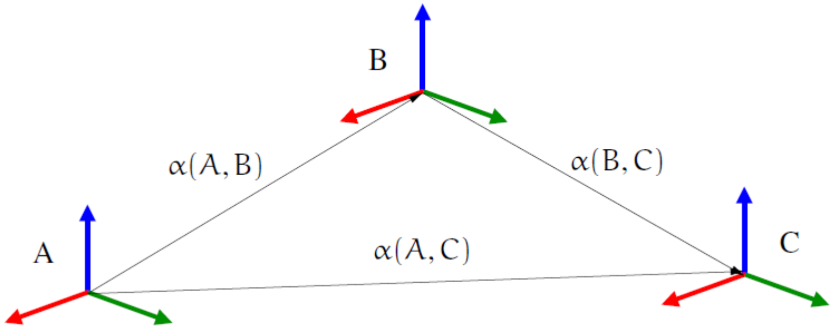
Different choices for frame H only change the expression for the Coriolis term.

# Propagating spatial accelerations

*Can accumulate spatial accelerations across multiple frames*

Spatial velocity propagation relationship

$$\mathcal{V}(A,C) = \phi^*(B,C)\,\mathcal{V}(A,B) + \mathcal{V}(B,C)$$




$$\alpha_H(A,C) = \phi^*(B,C)\,\alpha_H(A,B) + \alpha_H(B,C) + \underbrace{\frac{d_H \phi^*(B,C)}{dt}\,\mathcal{V}(A,B)}_{\text{Extra Coriolis term "a"}}$$

Differentiating the velocity expression with respect to the H frame yields the following spatial acceleration propagation relationship:

# Lie Group theory connections

# SE3 Lie Group connections

- Rotations form the **SO3** Lie group
- Homogenous transforms form the **SE3** Lie group
  - Spatial velocities defined here are closely related to (but not the same as) left/right trivialization elements of the Lie algebra
  - The spatial cross product is the Lie bracket (commutator) operator
  - $\phi^*(y, x)$ corresponds to the **Ad** adjoint transformations for the SE3 Lie group
- We mention these connections for completeness, but these will not be essential to our development
- More on these connections in book appendix

# Spatial Inertia

# Rigid body inertias

- Mass properties of a rigid body are characterized by
  - Scalar mass, m
  - First moment of inertia 3-vector p (vector from the body frame to the CM)
  - Second moment of inertia, 3x3 inertia matrix J
- Traditionally these terms are kept apart in the linear and angular equations of motion
  - This works well only at CM
  - Elsewhere get nasty cross-coupling terms

# Parallel axis methods

- **Parallel axis theorem** allows one to transform inertia properties from one body reference point to another
- Plain 3x3 rigid body inertia from CM inertia

$$\mathcal{J}(x) = \mathcal{J}(C) - m\,\tilde{p}(x)\,\tilde{p}(x)$$

- Inertia transformation from arbitrary point y to point x is more involved

# Spatial Inertia at CM

Body kinetic energy can be defined by the linear and angular terms at the CM

*linear* *angular*

$$\mathfrak{K}_e = \frac{1}{2}\mathfrak{m}v^2(\mathbb{C}) + \frac{1}{2}\omega^*(\mathbb{C})\,\mathcal{J}(\mathbb{C})\omega(\mathbb{C})$$

$$= \frac{1}{2}\mathcal{V}^*(\mathbb{C})\begin{pmatrix} \mathcal{J}(\mathbb{C}) & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}I_3 \end{pmatrix}\mathcal{V}(\mathbb{C})$$

$$= \frac{1}{2}\mathcal{V}^*(\mathbb{C})\,\underline{M(\mathbb{C})}\,\mathcal{V}(\mathbb{C})$$

$$M(\mathbb{C}) \triangleq \begin{pmatrix} \mathcal{J}(\mathbb{C}) & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}I_3 \end{pmatrix}$$

***spatial inertia*** *at the center of mass (6x6 matrix)*

# Spatial inertia – away from CM

## Kinetic energy is invariant to reference point

$$\mathfrak{K}_e = \frac{1}{2}\mathcal{V}^*(\mathbb{C})\, M(\mathbb{C})\, \mathcal{V}(\mathbb{C})$$
$$= \frac{1}{2}\overbrace{\mathcal{V}^*(x)\,\phi(x,\mathbb{C})}\, M(\mathbb{C})\, \overbrace{\phi^*(x,\mathbb{C})\,\mathcal{V}(x)}$$
$$= \frac{1}{2}\mathcal{V}^*(x)\, M(x)\, \mathcal{V}(x)$$

$$\mathcal{V}(\mathbb{C}) = \phi^*(x,\mathbb{C})\,\mathcal{V}(x)$$

*rigid body transf. of spatial velocity*

$$M(x) = \begin{pmatrix} \mathcal{J}(x) & m\widetilde{p}(x) \\ -m\widetilde{p}(x) & mI_3 \end{pmatrix}$$

***spatial inertia** at an arbitrary point (6x6 matrix)*

# Structure of the spatial inertia

$$\mathcal{M}(x) = \phi(x,\mathbb{C})\, \mathcal{M}(\mathbb{C})\, \phi^*(x,\mathbb{C})$$

$$= \begin{pmatrix} I & \tilde{\ell}(x,\mathbb{C}) \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathcal{J}(\mathbb{C}) & \mathbf{0} \\ \mathbf{0} & m I_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\tilde{\ell}(x,\mathbb{C}) & I \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{J}(\mathbb{C}) - m\,\tilde{\ell}(x,\mathbb{C})\,\tilde{\ell}(x,\mathbb{C}) & m\,\tilde{\ell}(x,\mathbb{C}) \\ -m\,\tilde{\ell}(x,\mathbb{C}) & m I_3 \end{pmatrix}$$

*parallel axis theorem for rigid body inertias*

*first moment of inertia*

*mass*

The spatial inertia matrix is always **symmetric** and **non-negative definite**

# Parallel axis theorem for spatial inertias

*Would like to move spatial inertia from one reference point to another*

**Claim:** $$M(y) = \phi(y,x)\, M(x)\, \phi^*(y,x)$$

parallel axis theorem for spatial inertias

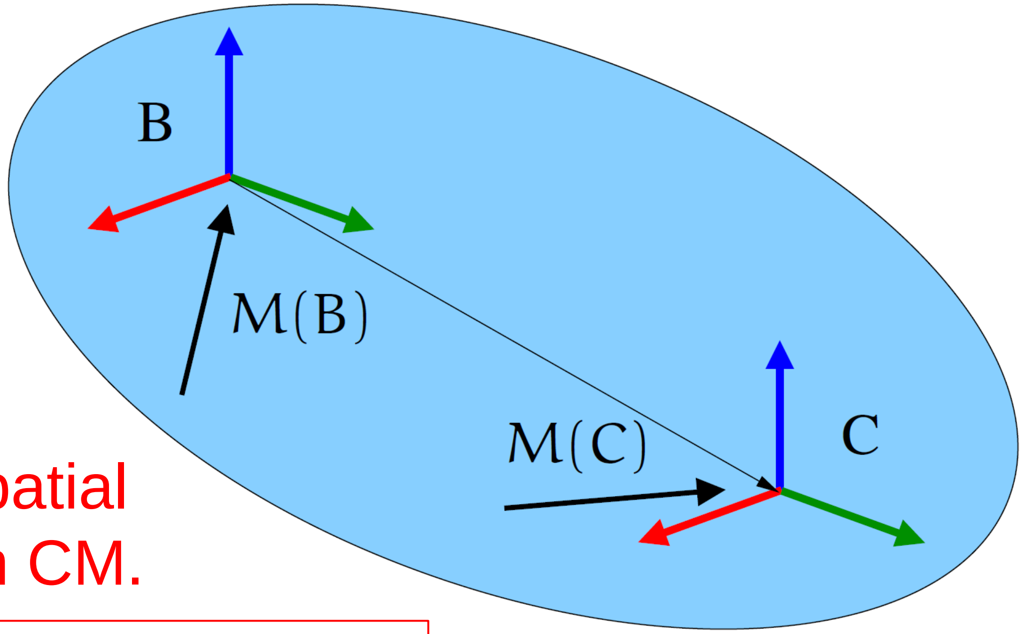
**Verification:**

$$\begin{aligned} M(y) &= \phi(y,\mathbb{C})\, M(\mathbb{C})\, \phi^*(y,\mathbb{C}) \\ &= \phi(y,x)\, \phi(x,\mathbb{C})\, M(\mathbb{C})\, \phi^*(x,\mathbb{C})\phi^*(y,x) \\ &= \phi(y,x)\, M(x)\, \phi^*(y,x) \end{aligned}$$

using $$\phi(y,\mathbb{C}) = \phi(y,x)\, \phi(x,\mathbb{C})$$

# Transforming Spatial Inertias

General way to transform spatial inertias across any pair of points on a rigid body




Parallel axis theorem for spatial inertias does not depend on CM.

$$^{B}M(B) = \phi(B,C)\ {}^{C}M(C)\ \phi^{*}(B,C)$$

# Rigid Body Kinetic Energy

- Kinetic energy is invariant to reference point when working with spatial quantities:

$$\mathfrak{K}_e = \frac{1}{2}\mathcal{V}^*(x)M(x)\mathcal{V}(x) = \frac{1}{2}\mathcal{V}^*(y)M(y)\mathcal{V}(y)$$

- This is a generalization of the well know quadratic expression for linear and angular energies at the CM

# Invariance of Kinetic Energy

Claim:

$$\mathfrak{K}_e = \frac{1}{2}\mathcal{V}^*(x)M(x)\mathcal{V}(x) = \frac{1}{2}\mathcal{V}^*(y)M(y)\mathcal{V}(y)$$

Verification:

$$\begin{aligned}\mathfrak{K}_e &= \frac{1}{2}\mathcal{V}^*(y)\ M(y)\ \mathcal{V}(y) \\ &= \frac{1}{2}\mathcal{V}^*(x)\ \phi(x,y)\ M(y)\ \phi^*(x,y)\ \mathcal{V}(x) \\ &= \frac{1}{2}\mathcal{V}^*(x)\ M(x)\ \mathcal{V}(x)\end{aligned}$$

# Accumulating spatial inertias

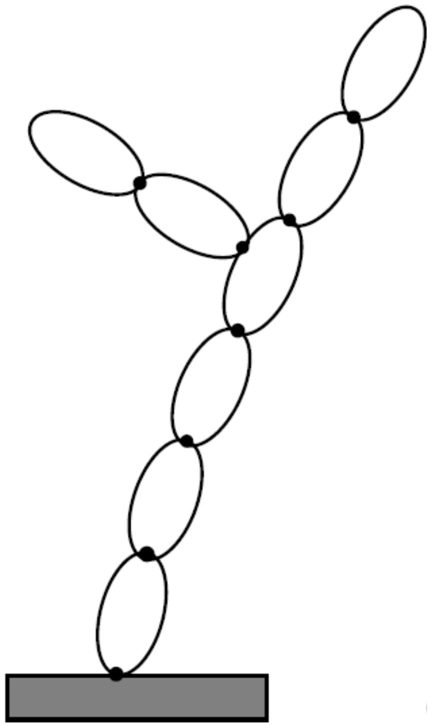
Often need total effective mass properties of a collection of bodies

- requires transforming all mass properties to a common point (parallel axis theorem) and then summing them up

$$^{B}M = \sum_{i} \phi(B,i)\, M(i)\, \phi^{*}(B,i)$$

*overall spatial inertia at B*

*transform spatial inertia to B*



# Spatial Momentum

# Rigid body spatial momentum

- About CM get standard form

$$\mathfrak{h}(C) \triangleq \begin{bmatrix} \mathcal{J}(C)\omega(C) \\ m v(C) \end{bmatrix} \overset{2.10}{=} M(C)\mathcal{V}(C)$$

spatial momentum

angular

linear

- Spatial momentum about point z

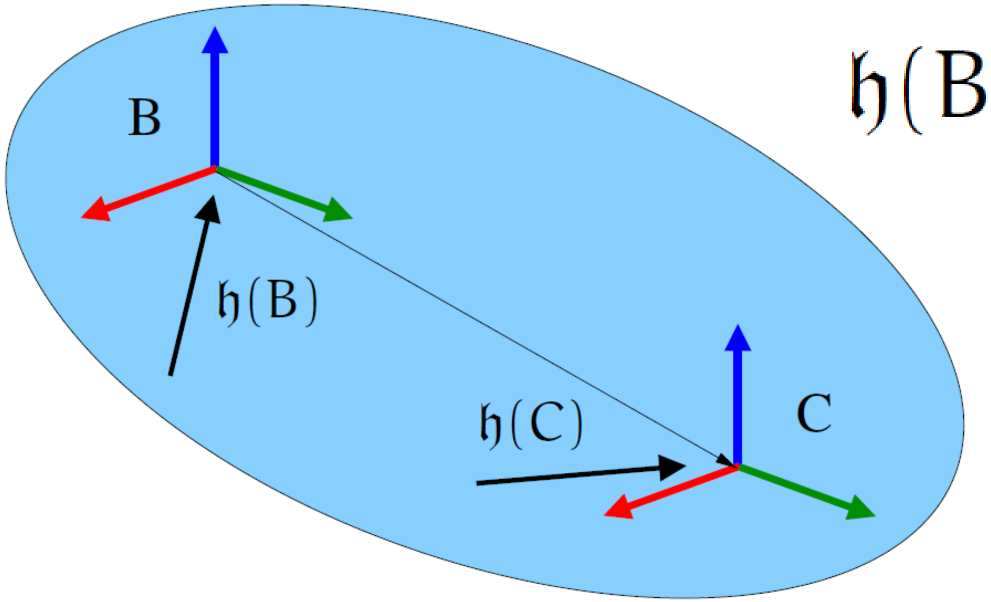
$$\mathfrak{h}(z) \triangleq M(z)\mathcal{V}(z) \quad \in \mathcal{R}^6$$

- Can transform from CM to another point via

$$\mathfrak{h}(x) = \phi(x, \mathbb{C})\, \mathfrak{h}(\mathbb{C})$$

# Transforming Spatial Momentum

Can transform spatial momentum across any pair of points




$$\mathfrak{h}(B) = \phi(B, C)\, \mathfrak{h}(C)$$

*(similar to transformation for spatial forces)*

# Spatial Transformations Recap

*Spatial notation offers concise & consistent transformation expressions for arbitrary non-CM points*

*rigid body transformation matrix*

Spatial velocities $\quad {}^{C}\mathcal{V}(A,C) = \phi^*(B,C)\,{}^{B}\mathcal{V}(A,B)$

Spatial forces $\quad {}^{B}\mathfrak{f}(B) = \phi(B,C)\,{}^{C}\mathfrak{f}(C)$

Spatial inertia $\quad M(x) = \phi(x,y)M(y)\phi^*(x,y)$

Kinetic energy $\quad \mathfrak{K}_e = \frac{1}{2}\mathcal{V}^*(x)M(x)\mathcal{V}(x) = \frac{1}{2}\mathcal{V}^*(y)M(y)\mathcal{V}(y)$

Spatial momentum $\quad \mathfrak{h}(x) = \phi(x,y)\mathfrak{h}(y)$

# Computational optimization

While the expressions are compact and concise, most of them involve sparse terms, and can optimize implementations for speed.

# SOA Foundations Track Topics (serial-chain rigid body systems)

1. **Spatial (6D) notation** – spatial velocities, forces, inertias; spatial cross-product, rigid body transformations & properties; parallel axis theorem
2. **Single rigid body dynamics** – equations of motion about arbitrary frame using spatial notation
3. **Serial-chain kinematics** – minimal coordinate formulation, hinges, velocity recursions, Jacobians; first spatial operators; O(N) scatter and gather recursions
4. **Serial-chain dynamics** – equations of motion using spatial operators; Newton–Euler mass matrix factorization; O(N) inverse dynamics
5. **Mass matrix -** composite rigid body inertia; forward Lyapunov equation; mass matrix decomposition; mass matrix computation; alternative inverse dynamics
6. **Articulated body inertia -** Concept and definition; Riccati equation; alternative force decompositions
7. **Mass matrix factorization and inversion** – spatial operator identities; Innovations factorization of the mass matrix; Inversion of the mass matrix
8. **Recursive forward dynamics** – O(N) recursive forward dynamics algorithm; including gravity and external forces; inter-body forces identity